$\boxed{2.6 \quad 11}$

$$\int_0^{2\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\, d\theta$$

$z = e^{i\theta} \qquad dz = iz\, d\theta$

$\frac{1}{2}(z^2+z^{-2}) = \frac{1}{2}(e^{2\theta i}+e^{-2\theta i}) = \cos(2\theta)$

$\frac{1}{2}(z+z^{-1}) = \cos(\theta)$

$$= \int_{|z|=1} \frac{\frac{1}{2}(z^2+z^{-2})}{(1-2a\frac{1}{2}(z+z^{-1})+a^2)\, iz}\, dz$$

$\boxed{-1<a<1}$

$$= \int_{|z|=1} \frac{z^4+1}{(2z^2-2az(z^2+1)+2a^2z^2)\, iz}\, dz$$

$$= \int_{|z|=1} \frac{z^4+1}{(2z-2a(z^2+1)+2a^2z)\, iz^2}\, dz$$

$$= \int_{|z|=1} \frac{z^4+1}{2iz^2(-az^2+(a^2+1)z-a)}\, dz$$

$$= \frac{i}{2}\int_{|z|=1} \frac{z^4+1}{z^2(az^2-(a^2+1)z+a)}\, dz$$

$$f(z) = \frac{z^4+1}{z^2(az^2-(a^2+1)z+a)}$$

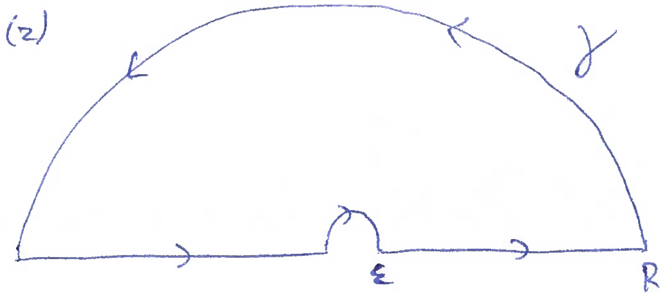
$$= \frac{i}{2}\cdot 2\pi i\left(\mathrm{Res}(f;0)+\mathrm{Res}(f;a)\right)$$

$$= -\pi\left(\frac{a^2+1}{a^2}+\frac{a^4+1}{a^2(a^2-1)}\right) = -\frac{\pi}{a^2}\,\frac{(a^2+1)(a^2-1)+a^4+1}{a^2-1}$$

$$= \frac{2\pi a^2}{1-a^2}$$

26 $\boxed{17}$

$$f(z) = \frac{\log(z)}{1+z^2} \qquad \log(z) = \ln|z| + i\arg(z)$$



$0 < \varepsilon < 1 < R$

Singularity: $z = i$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \operatorname{Res}(f; i)$$

$$= \frac{\log(i)}{2i} = \frac{\frac{\pi}{2} i}{2i} = \frac{\pi}{4}$$

$|z| = R.$ $\quad |f(z)| \le \dfrac{2\log(R)}{R^2-1}$

$$\left|\int_{\substack{|z|=R\\ \mathrm{Im}>0}} f(z)\,dz\right| \le \pi R \cdot \frac{2\log(R)}{R^2-1} \to 0$$

$|z| = \varepsilon.$ $\quad |f(z)| \le \dfrac{-2\log(\varepsilon)}{1-\varepsilon^2}$

$$\left|\int_{\substack{|z|=\varepsilon\\ \mathrm{Im}>0}} f(z)\,dz\right| \le \pi\varepsilon \cdot \frac{-2\log(\varepsilon)}{1-\varepsilon^2} \to 0$$

$$\int_{-\infty}^{0} \frac{\log(-x) + i\pi}{1+x^2}\,dx + \int_0^\infty \frac{\log(x)}{1+x^2}\,dx = \frac{\pi}{4}\cdot 2\pi i$$

$$\int_0^\infty \frac{\log(x)}{1+x^2}\,dx = 0 \qquad \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

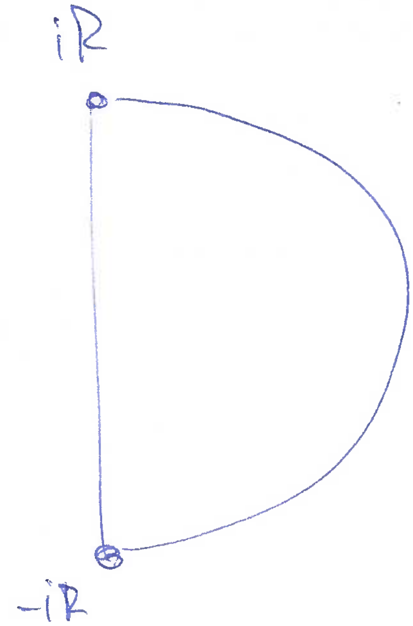
$$z^4 - 5z^2 + 3 = e^{-z}$$

$$|e^{-iy}| = 1.$$

$\boxed{3.1 \ \#9}$

$$(iy)^4 - 5(iy)^2 + 3 =$$

$$|y^4 + 5y^2 + 3| > |e^{-iy}|$$



$$f(z) = z^4 - 5z^2 + 3 - e^{-z}$$

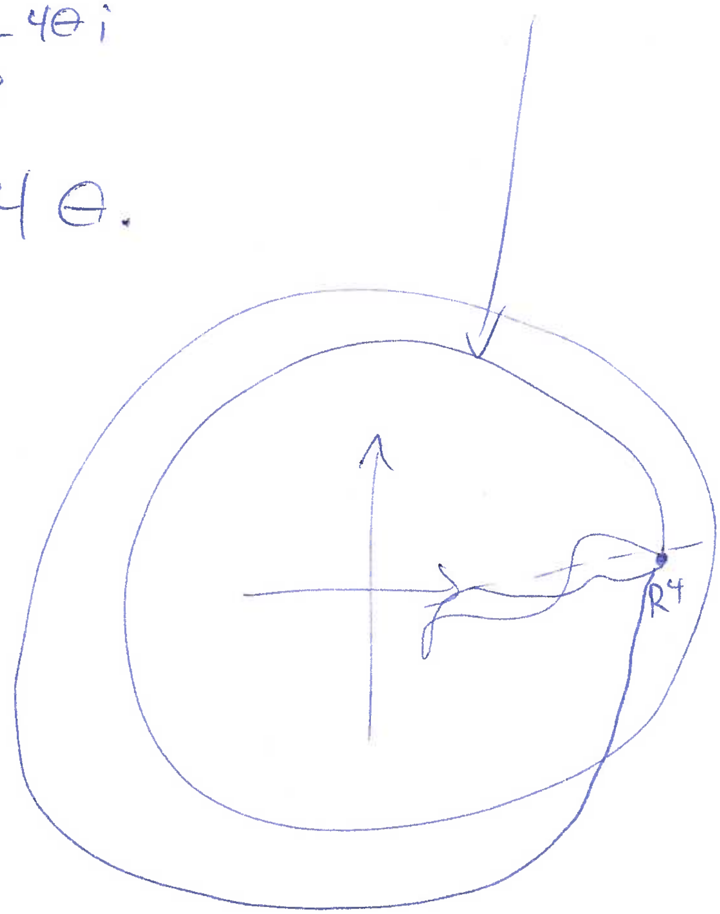
$$f(iy) = y^4 + 5y^2 + 3 \cancel{4} - e^{-iy}$$

$$\mathrm{Re}(f(iy)) > 0.$$

$$f(Re^{i\theta}) \approx R^4 e^{-4\theta i}$$

$$\arg(f(Re^{i\theta})) \approx 4\theta.$$

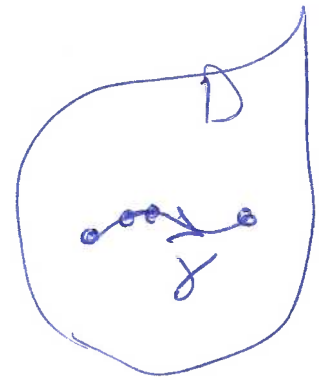
2 roots in right half-plane.

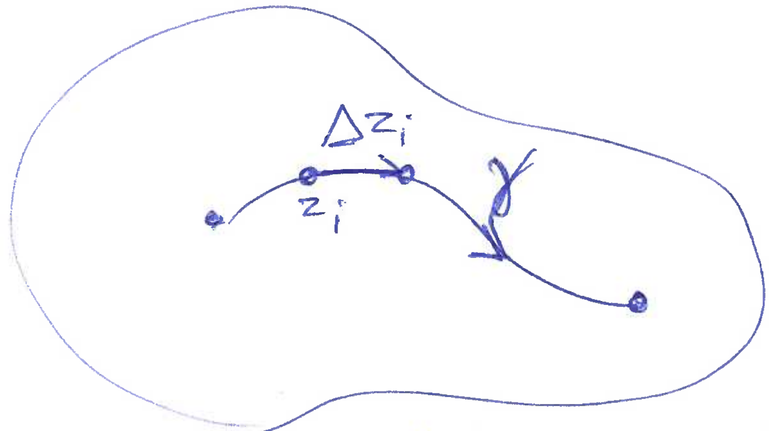


$\log((1-i)^4)$

$f : D \longrightarrow \mathbb{C}$ cont. $D \subseteq \mathbb{C}$ open.

$\gamma \subseteq D$ oriented curve.





$$\int_\gamma f(z)\,dz = \lim_{\Delta z \to 0} \sum f(z_i)\Delta z_i$$

$\gamma : [a,b] \longrightarrow \mathbb{C}$ parametrization:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

First: $g : [a,b] \longrightarrow \mathbb{C}$ continuous.

$$\int_a^b g(t)\,dt = \int_a^b \mathrm{Re}(g(t))\,dt + i\int_a^b \mathrm{Im}(g(t))\,dt$$

And: $\gamma(t) = x(t) + i\,y(t)$. $\gamma'(t) = x'(t) + i\,y'(t)$

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt.$$

$$\Theta = \operatorname{Arg}\left(\int_a^b g(t)\,dt\right)$$

$$\left|\int_a^b g(t)\,dt\right| = e^{-i\Theta}\int_a^b g(t)\,dt$$

$$= \int_a^b e^{-i\Theta} g(t)\,dt \qquad \Leftarrow \text{real.}$$

$$= \int_a^b \operatorname{Re}\left(e^{-it} g(t)\right)dt$$

$$\le \int_a^b \left|e^{-it} g(t)\right|dt$$

$$= \int_a^b |g(t)|\,dt.$$

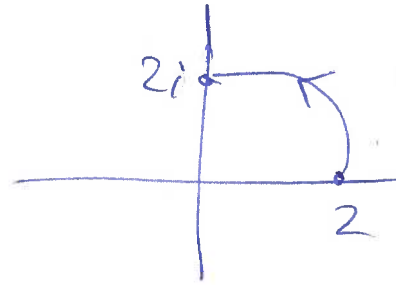
$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right|$$

$$\le \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt$$

$$\le \left(\max_{z\in\gamma}|f(z)|\right)\cdot\underbrace{\int_a^b |\gamma'(t)|\,dt}_{\text{length}(\gamma)}.$$

3

$\gamma(t) = 2e^{it}, \quad 0 \le t \le \pi/2$



$$\int_\gamma \left(z^2 - 3|z| + \operatorname{Im} z\right) dz$$

$$= \int_0^{\pi/2} \left(4e^{2it} - 3\cdot 2 + 2\sin(t)\right) \cancel{dt}\; 2ie^{it}\, dt$$

$$= \int_0^{\pi/2} \left(8ie^{3it} - 12ie^{it} + 4i\sin(t)\,e^{it}\right) dt$$

$$= \cancel{\left[\frac{8}{3}e^{3it} - 12e^{it} + \frac{1}{i}e^{it} - 2t\right]_0^{\pi/2}}$$

$$= \int_0^{\pi/2} \left(8ie^{3it} - 12ie^{it} + \frac{4i}{2i}e^{it}\left(e^{it} - e^{-it}\right)\right) dt$$

$$= \int_0^{\pi/2} \left(8ie^{3it} - 12ie^{it} + 2e^{2it} - 2\right) dt$$

$$= \left[\frac{8}{3}e^{3it} - 12e^{it} - ie^{2it} - 2t\right]_0^{\pi/2}$$

$$= \left(-\frac{8i}{3} - 12i + i - \pi\right) - \left(\frac{8}{3} - 12 - i\right)$$

$$= \frac{28}{3} - \pi - \frac{38}{3}i$$

$$\int_{-\pi/2}^{\pi} \cos(t+i)\,dt = \int_{-\pi/2}^{\pi} \Big(\cos(t)\cosh(1) - i\sin(t)\sinh(1)\Big)\,dt$$

$$= \Big[\sin(t)\cosh(1) + i\cos(t)\sinh(1)\Big]_{-\pi/2}^{\pi}$$

$$= -i\sinh(1) + \cosh(1).$$

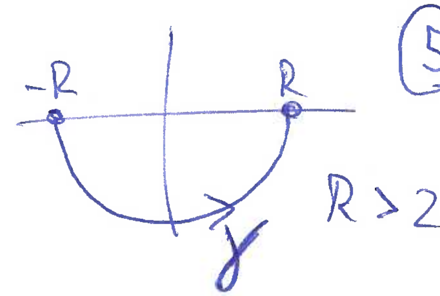
Example:

Find $\int_\gamma \cos(z)\,dz$.

Example

Estimate

$$\left| \int_\gamma \frac{1}{z^2+4}\, dz \right|$$



$\gamma(t) = Re^{it}$

$-\pi \le t \le 0.$

$|z| = R > 2$

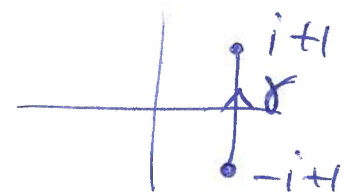
$|z^2| = R^2 > 4$

$|z^2+4| \geq R^2 - 4$

$\left|\frac{1}{z^2+4}\right| \leq \frac{1}{R^2-4}$ for all $z \in \gamma$

$$\left| \int_\gamma \frac{1}{z^2+4}\, dz \right| \leq \frac{1}{R^2-4} \cdot \text{length}(\gamma) = \frac{\pi R}{R^2-4}$$

Example



$\gamma(t) = 1 + it$

$-1 \le t \le 1$

$$\left| \int_\gamma e^{-z}\, dz \right| \leq \cancel{\text{length}(\gamma) = 2}$$

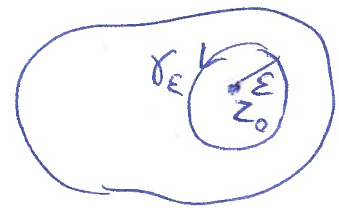
$\text{Re}(z) = 1$ for $z \in \gamma$.

$|e^{-z}| = e^{-1}$

$$\left| \int_\gamma e^{-z}\, dz \right| \leq e^{-1}\, \text{length}(\gamma) = \frac{2}{e}.$$

Example $f : D \xrightarrow{\text{cont}} \mathbb{C}$, D open. $z_0 \in D$.

$B(z_0, \varepsilon) \subseteq D$.



Show $$\boxed{\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)\,dz}{z - z_0} = f(z_0).}$$

$$\gamma_\varepsilon(t) = z_0 + \varepsilon e^{it}, \quad 0 \le t \le 2\pi$$

$$\gamma_\varepsilon'(t) = i\varepsilon e^{it}$$

$$\int_{\gamma_\varepsilon} \frac{f(z)\,dz}{z - z_0} = \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{it})}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt$$

$$= i \int_0^{2\pi} f(z_0 + \varepsilon e^{it})\,dt$$

$$\left| \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}\,dz - f(z_0) \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \left(f(z_0 + \varepsilon e^{it}) - f(z_0) \right) dt \right|$$

$$\cancel{\le \frac{1}{2\pi} \left(\max_{z \in \gamma_\varepsilon} |f(z) - f(z_0)| \right) \text{length}}$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + \varepsilon e^{it}) - f(z_0) \right| dt$$

$$\le \max_{z \in \gamma} |f(z) - f(z_0)| \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

PROP $f, g : [a,b] \longrightarrow \mathbb{C}$ differentiable

Then fg diff and

$$(fg)'(t) = f'(t)\,g(t) + f(t)\,g'(t).$$

Proof

$$f(t) = f_1(t) + i f_2(t)$$
$$g(t) = g_1(t) + i g_2(t).$$
$$(fg)(t) = \left(f_1(t)\,g_1(t) - f_2(t)\,g_2(t)\right) + i\left(f_1(t)\,g_2(t) + f_2(t)\,g_1(t)\right)$$
$$(fg)' = \left((f_1 g_1)' - (f_2 g_2)'\right) + i\left((f_1 g_2)' + (f_2 g_1)'\right)$$
$$= \underline{f_1' g_1} + \underline{f_1 g_1'} - \underline{f_2' g_2} - f_2 g_2'$$
$$+ i\left(\underline{f_1' g_2} + \underline{f_1 g_2'} + \underline{f_2' g_1} + \underline{f_2 g_1'}\right)$$
$$= f_1' g_1 + f_1' i g_2 + i f_2' g_1 + i f_2' i g_2$$
$$+ f_1 g_1' + f_1 i g_2' + i f_2 g_1' + i f_2 i g_2'$$
$$= (f_1' + i f_2')(g_1 + i g_2) + (f_1 + i f_2)(g_1' + i g_2')$$
$$= f' g + f g'$$

Example $f : \mathbb{R} \to \mathbb{C}$.

$$\frac{d}{dt}\left(f(t)^n\right) = n\, f(t)^{n-1} f'(t)$$

for all $n \in \mathbb{Z}$.

True for $n = 0$.

$$\frac{d}{dt}\left(f(t)^{n+1}\right) = \left(\frac{d}{dt} f(t)^n\right) \cdot f(t) + f(t)^n f'(t)$$

Shows: True for $n \Leftrightarrow$ True for $n+1$

($\Leftarrow$ requires $f(t) \neq 0$.)

<u>Example</u> $f : \mathbb{R} \to \mathbb{C}$.

$$\frac{d}{dt}\left(f(t)^n\right) = n\, f(t)^{n-1} f'(t) \quad \text{for all } n \in \mathbb{Z}.$$

<u>Example</u>

~~for~~ $\gamma : [a,b] \to \mathbb{C}$.

Show $\displaystyle\int_\gamma z^m\, dz = \frac{1}{m+1}\left(z_2^{m+1} - z_1^{m+1}\right)$.

$$\int_\gamma z^m\, dz = \int_a^b \gamma(t)^m \gamma'(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\left(\frac{1}{m+1}\gamma(t)^{m+1}\right) dt$$

$$= \left[\frac{1}{m+1}\gamma(t)^{m+1}\right]_a^b = \frac{1}{m+1}\left(\gamma(b)^{m+1} - \gamma(a)^{m+1}\right)$$

$f : D \to \mathbb{C}$ $\qquad f(z) = p(z) + i\,q(z)$.

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x} p(x+iy), \ldots, \quad \frac{\partial q}{\partial x} = \frac{\partial}{\partial x} q(x+iy)$$

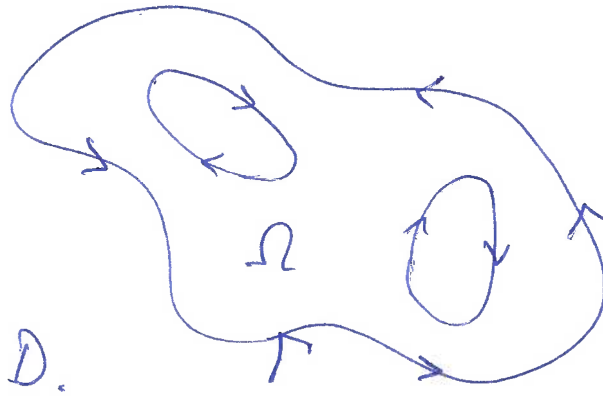
$$\boxed{\frac{\partial f}{\partial x} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x}} \qquad \boxed{\frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + i\frac{\partial q}{\partial y}}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y} p(x+iy) \qquad \frac{\partial q}{\partial y} = \frac{\partial}{\partial y} q(x+iy)$$

Note: $\displaystyle i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = i\frac{\partial p}{\partial x} - \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} - i\frac{\partial q}{\partial y}$

$$= -\left(\frac{\partial q}{\partial x} + \frac{\partial p}{\partial y}\right) + i\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right)$$

Green's Thm

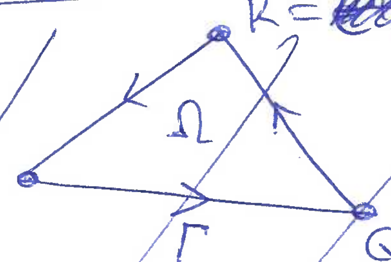


$f : D \to \mathbb{C}$, D open,

$\overline{\Omega} \subseteq D$, f has cont. partial derivatives on D.

Then $$\int_\Gamma f(z)\,dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) dx\,dy$$

Triangle



PQ : $\gamma_1(t) = (1-t)(a+ib) + t(c+id)$, $0 \le t \le 1$

QR : $\gamma_2(t) = (1-t)(c+id) + t(r+is)$, $0 \le t \le 1$

RP : $\gamma_3(t) = (1-t)(r+is) + t(a+ib)$, $0 \le t \le 1$

$\gamma(t) = x(t) + i y(t)$.

$f(z) = p(z) + i q(z)$

$$\operatorname{Re} \int_\Gamma f(z)\,dz = \operatorname{Re} \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \left(p(\gamma(t))\,x'(t) - q(\gamma(t))\,y'(t) \right) dt$$

$$\operatorname{Re}\ \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$$

Green $\Leftrightarrow$

$\vec{F} = (u, v)$ vector field on Ω.

$\gamma(t) = (x(t), y(t))$ parametrizes $\partial\Omega = \Gamma$. $a \le t \le b$.

$f(z) = \overline{F} = u(z) - i\,v(z)$.

Green's Thm: $\displaystyle\int_\gamma f(z)\,dz = \iint_\Omega i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$

$$\int_a^b \big(u(\gamma(t)) - i v(\gamma(t))\big)\big(x'(t) + i y'(t)\big)\,dt = \iint_\Omega \left(\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) dx\,dy$$

Real parts:

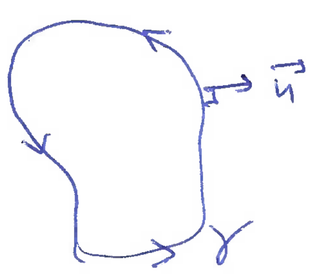
$$\int_a^b (u, v)\cdot(x', y')\,dt = \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\int_\gamma \vec{F}\cdot\gamma'(t)\,dt = \iint_\Omega \operatorname{curl}(F)\,dx\,dy$$

Im part:

$$\int_a^b \big(u(\gamma(t))\,y'(t) - v(\gamma(t))\,x'(t)\big)\,dt = \iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy$$

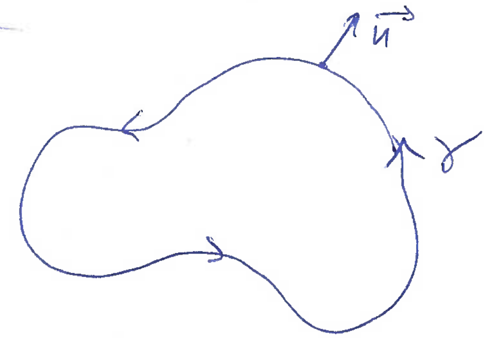
$$\int_\gamma \vec{F}\cdot\vec{n}\,ds = \iint \operatorname{div}(\vec{F})\,dx\,dy$$



Green's Thm Divergence

$\vec{F}$ vector field

$\vec{F} = (u, v)$



$$\int_\gamma \vec{F} \cdot \vec{n}\, ds = \int_a^b (u,v) \cdot (y', -x')\, dt \qquad \gamma(t) = (x(t), y(t))$$

$$= \int_a^b \left(u(t)\, \cancel{x}y'(t) - v(t)\, x'(t) \right) dt$$

$$\underset{\substack{\text{Green} \\ \text{Div}}}{=} \iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\, dy$$

Green's Thm Curl

$$\int_\gamma \vec{F} \cdot \vec{v}\, ds = \int_a^b (u,v) \cdot (x', y')\, dt = \int_a^b \left(u(t)\, x'(t) + v(t)\, y'(t) \right) dt$$

$$\underset{\substack{\text{Green} \\ \text{Curl}}}{=} \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

Equivalent: Recover curl:

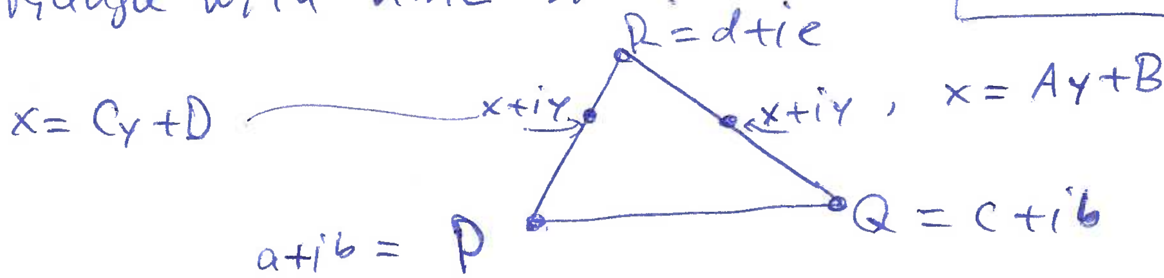
$$\cancel{\int \vec{F} \cdot \vec{v}\, ds} \quad \int_a^b \left(u(t)\, x'(t) + v(t)\, y'(t) \right) dt = \int_a^b \left| v(t)\, y'(t) - (-u(t))\, x'(t) \right| dt$$

$$\underset{\substack{\text{Green} \\ \text{Div}}}{=} \iint_\Omega \left(\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}(-u(t)) \right) dx\, dy$$

$$= \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

Triangle with horiz side.

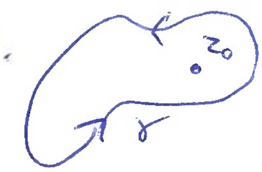
$\boxed{\text{Check Green's Thm}}$



$$A = \frac{c-d}{b-e}, \quad B = c - bA$$

$$C = \frac{d-a}{e-b}, \quad D = d - eC$$

Example γ simple closed loop, pos orient.



$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} = \begin{cases} 1 & \text{if } z_0 \text{ inside } \gamma \\ 0 & \text{else.} \end{cases}$$

Proof

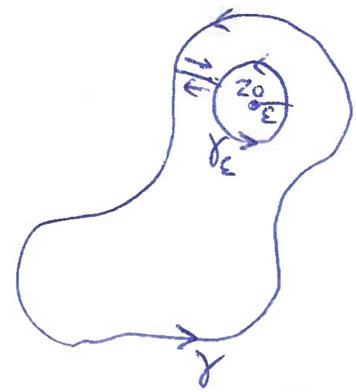
$$f(z) = \frac{1}{z-z_0}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x+iy) = -\frac{1}{(z-z_0)^2}\frac{\partial}{\partial x}(z-z_0) = \frac{-1}{(z-z_0)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x+iy) = -(z-z_0)^{-2}\frac{\partial}{\partial y}(z-z_0) = \frac{-i}{(z-z_0)^2}$$

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

$\boxed{\text{Assume inside}}$



Green's Thm:

$$\int_\gamma \frac{dz}{z-z_0} = \int_{\gamma_\varepsilon} \frac{dz}{z-z_0}$$

$$\gamma_\varepsilon(t) = z_0 + \varepsilon e^{it}$$

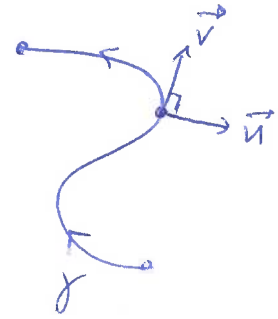
$$= \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt$$

$$= \int_0^{2\pi} i\, dt = 2\pi i$$

Green's formula

$D \subseteq \mathbb{C}$ open, $f : D \longrightarrow \mathbb{R}$ diff.

$\gamma \subseteq D$ oriented curve.



$$\frac{\partial f}{\partial s} = \vec{\nabla} f \cdot \vec{v} \qquad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial n} = \vec{\nabla} f \cdot \vec{n} \qquad \vec{v} = \frac{(x', y')}{\sqrt{x'^2 + y'^2}}$$

if $\gamma : [a,b] \longrightarrow D$ parametrization.

$$\vec{n} = \frac{(y', -x')}{\sqrt{x'^2 + y'^2}} \qquad ds = \sqrt{x'^2 + y'^2}\, dt$$

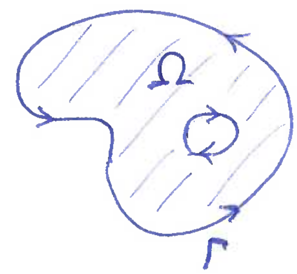
Divergence Thm

$$\int_\Gamma \vec{F} \cdot \vec{n}\, ds = \iint_\Omega \operatorname{div}(\vec{F})\, dx\, dy$$

$$\vec{F} = (u, v)$$

$$\operatorname{div}(\vec{F}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$\int_a^b (u y' - v x')\, dt = \iint_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy$$



Green's formula:

$$\int_\Gamma \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n}\right) ds = \int_\Gamma \left(g \vec{\nabla f} - f \vec{\nabla g}\right) \cdot \vec{n}\, ds$$

$$= \iint_\Omega \operatorname{div}\left(g \vec{\nabla f} - f \vec{\nabla g}\right) dx\, dy = \iint_\Omega \left[\frac{\partial}{\partial x}\left(g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(g \frac{\partial f}{\partial y} - f \frac{\partial g}{\partial y}\right)\right]$$

$$= \iint_\Omega (g \Delta f - f \Delta g)\, dx\, dy$$

Laplacian: $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

Example

$$\int_\Gamma f \frac{\partial f}{\partial n}\, ds = \int_\Gamma f \vec{\nabla} f \cdot \vec{n}\, ds = \iint_\Omega \operatorname{div}(f \vec{\nabla} f)\, dx\, dy$$

$$= \iint_\Omega \left[\frac{\partial}{\partial x}\left(f \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \frac{\partial f}{\partial y}\right)\right] dx\, dy$$

$$= \iint_\Omega \left[f \Delta f + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dx\, dy$$

Def $f : D \longrightarrow \mathbb{R}$ is harmonic if $\Delta f = 0$.

Thm $f : D \longrightarrow \mathbb{R}$ harmonic, $\overline{\Omega} \subseteq D$, $f(z) = 0$ for $z \in \Gamma = \partial \Omega$.
Then $f(z) = 0$ $\forall z \in \Omega$.

Proof

$$0 = \int_\Gamma f \frac{\partial f}{\partial n}\, ds = \iint_\Omega \left[0 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dx\, dy$$

Shows $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ for $z \in \Omega$.

$\square$