Intersection multiplicities

$i : X \hookrightarrow Y$ regular embedding of codim. d.

$f : V \longrightarrow Y$ morphism.

V of pure dim. k.

$$\begin{tikzcd} C = C_W V \subseteq N \arrow[r] \arrow[d] & W \arrow[l, bend right, "s"'] \arrow[r, "j"] \arrow[d, "g"] & V \arrow[d, "f"] \\ N_X Y \arrow[r] & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$$X \cdot V = s^*([C]) \in A_{k-d}(W).$$

Let $Z \subseteq W$ be a proper component:

$$\dim(Z) = k - d.$$

<u>Def</u> $i(Z, X \cdot V ; Y) =$ coeff. of $[Z]$ in $X \cdot V$.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & V \arrow[d] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

V pure dim. k.

i regular of codim d.

<u>Prop</u> $Z \subseteq W$ proper component.

(a) $1 \leq i(Z, X \cdot V; Y) \leq \operatorname{length}(\mathcal{O}_{Z,W})$.

(b) If $I(W) \subseteq \mathcal{O}_{Z,V}$ generated by regular seq. of length d, then

$$i(Z, X \cdot V; Y) = \operatorname{length}(\mathcal{O}_{Z,W}).$$

(c) If $\mathcal{O}_{Z,V}$ is Cohen-Macaulay, then (b) applies. In fact: $I(W)$ gen. by eqns. of $X \subseteq Y$, which is a regular sequence.

Proof

Recall: $C_W V$ has pure dim. k and

$C_W V \twoheadrightarrow W$ surjective.

$\exists$ irred. comp. $C' \subseteq C = C_W V$
such that $C' \twoheadrightarrow Z$.

Set $N_2 = N/_2$.

$$\begin{tikzcd}
C' \arrow[d] & \subseteq & N_2 \arrow[r] \arrow[d] & Z \arrow[d] & \\
C & \subseteq & N \arrow[r, two heads] & W \arrow[l, bend right, "s"'] \arrow[r] \arrow[d] & V \arrow[d] \\
& & N_X Y \arrow[r] \arrow[from=u, d] & X \arrow[r, hook, "i"'] & Y
\end{tikzcd}$$

$C' \subseteq N_2$ closed subvariety.

$\dim(C') = \dim(N_2) = k$.

$\Rightarrow C' = N_2$

$\Rightarrow s^*([C']) = [Z]$.

$i(Z, X\cdot V; Y) = \text{coeff. of } [Z] \text{ in } s^*([C])$

$= \text{coeff. of } [C'] \text{ in } [C]$

$= \text{length}(\mathcal{O}_{C',C}).$

Since $C \subseteq N$ closed subscheme:

$$1 \leq \text{length}(\mathcal{O}_{C',C}) \leq \text{length}(\mathcal{O}_{N_Z,N})$$
$$= \text{length}(\mathcal{O}_{Z,W}).$$

Assume $I(W) \subseteq \mathcal{O}_{Z,V}$ gen. by regular seq. of length d.

Replace V with open subscheme meeting Z:

WLOG: $W \hookrightarrow V$ regular of codim. d.

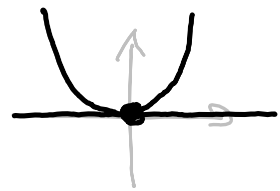
$$\begin{tikzcd} W \arrow[r, hook, "i'"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$X\cdot V = i^![V] = i'^*[V] = [W].$

$\square$

Example

$Y = \mathbb{A}^2$, $X = V(y) \subseteq \mathbb{A}^2$, $V = V(y - x^2) \subseteq \mathbb{A}^2$.

$W =$ 

$Z = (0,0) \in W$ proper component.

$i(Z, X \cdot V; \mathbb{A}^2) = 2.$

$$\begin{tikzcd}
C' \arrow[r, phantom, "="] \arrow[d] & N_Z = Z \times \mathbb{A}^1 \arrow[r] \arrow[d] & Z \arrow[d] & \\
C \arrow[r, phantom, "="] & N = W \times \mathbb{A}^1 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & V \arrow[d] \\
 & N_X Y = X \times \mathbb{A}^1 \arrow[r] & X \arrow[r, hook, "i"'] & \mathbb{A}^2
\end{tikzcd}$$

$$\mathcal{O}_{C',C} = \left(K[x,y]/\langle x^2 \rangle\right)_{\langle x \rangle} = K(y)[x]/\langle x^2 \rangle$$

$\text{length}(\mathcal{O}_{C',C}) = 2.$

<u>Example</u> $Y = \mathbb{A}^4$.

$X = V(x_1 - x_3,\ x_2 - x_4) \subseteq \mathbb{A}^4$.

$V = V(x_1x_3,\ x_2x_3,\ x_1x_4,\ x_2x_4) \subseteq \mathbb{A}^4$

$i : X \hookrightarrow \mathbb{A}^4$ regular of codim. 2.

$W = X \cap V$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$\mathcal{O}_W = K[x_1, x_2] / \langle x_1^2,\ x_1x_2,\ x_2^2 \rangle$

$Z = (0,0,0,0) \in W$ proper component.

$\mathrm{length}(\mathcal{O}_{Z,W}) = \mathrm{length}(\mathcal{O}_W) = 3$.

$V = V(x_1, x_2) \cup V(x_3, x_4)$

$\mathcal{O}_{V(x_1,x_2),V} = K(x_3, x_4)$ field.

$[V] = [V(x_1, x_2)] + [V(x_3, x_4)]$

$X \cdot V = i^![V] = i^![V(x_1,x_2)] + i^![V(x_3,x_4)]$

$= 2[Z]$.

$i(Z, X \cdot V; \mathbb{A}^4) = 2 < \mathrm{length}(\mathcal{O}_{Z,W}) = 3$.

Transversality

Bertini's Theorem

$k = \bar{k}$. $X \subseteq \mathbb{P}^n$ non-singular closed subvar.
$\exists$ hyperplane $H \subseteq \mathbb{P}^n$ such that
$X \nsubseteq H$ and $X \cap H$ is a (disjoint union of)
non-singular varieties.

In fact: $X \cap H$ is connected if $\dim(X) \geq 2$:

Thm X irred. projective variety$/k = \bar{k}$,
$\dim(X) \geq 2$, $D \subseteq X$ ample effective Cartier
divisor. Then D is connected.

In fact: $X \cap H$ is non-singular for
almost all hyperplanes $H \subseteq \mathbb{P}^n$.

$G = GL(n+1, k)$ acts transitively on $\mathbb{P}^n$.
G acts transitively on set of all
hyperplanes in X.

$\exists\, \emptyset \neq U \subseteq G$ open: $g.V(x_0) \cap X$ is
non-singular $\forall g \in U$.

Assume $K = \overline{K}$.

<u>Algebraic group</u>: Variety G with morphisms

$$\mu : G \times G \longrightarrow G \quad , \quad \iota : G \longrightarrow G$$

and a point $1 \in G$ which give G a group structure: $g \cdot h = \mu(g,h)$, $g^{-1} = \iota(g)$.

<u>G-variety</u>, $G \circlearrowright X$:

Variety X with morphism $a : G \times X \longrightarrow X$ that defines an action of G on X: $g \cdot x = a(g,x)$.

<u>Note</u>: If $G \circlearrowright X$ is transitive, then X is non-singular.

Any alg. group is non-singular.

<u>G-equivariant morphism</u>:

X, Y G-varieties, $f : X \to Y$ morphism.

f is G-equiv. if $f(g.x) = g.f(x)$.

Kleiman's Theorem (non-relative)

G algebraic group.

X transitive G-variety.

$Y, Z \subseteq X$ locally closed subvarieties.

For $s \in G$, $s.Y = \{s.y \mid y \in Y\} \subseteq X$.

(a) $\exists$ dense open $U \subseteq G$:

$\forall s \in U$: $s.Y \cap Z$ is a proper intersection.

(pure dim $= \dim Y + \dim Z - \dim X$.)

(b) Assume $\mathrm{char}(K) = 0$, Y, Z non-singular. Then:

$\exists$ dense open $U \subseteq G$:

$\forall s \in U$: $s.Y \cap Z$ is non-singular.

Example

$G \circlearrowright X$ transitive,

$Y, Z \subseteq X$ subvarieties,

$\dim(Y) + \dim(Z) = \dim(X)$. Then:

$s.Y \cap Z$ is finite for all s in dense open $U \subseteq G$.

Assume $\mathrm{char}(K) = 0$. Then:

For all s in dense open $U \subseteq G$:

$s.Y \cap Z$ is transversal.

(finite + reduced, each point $p \in s.Y \cap Z$ is non-sing. in $s.Y$ and in Z.)

Reason: $\dim(Y_{sing}) < \dim(Y)$

$s.Y_{sing} \cap Z = \emptyset \quad \forall s \in$ dense open $\subseteq G$.

$Y^\circ = Y - Y_{sing}, \quad Z^\circ = Z - Z_{sing}$.

$s.Y \cap Z = s.Y^\circ \cap Z^\circ$ transversal $\forall s \in$ dense open $\subseteq G$.