Smooth morphisms

$f: X \to Y$ morphism of alg. schemes $/K$.

Flat: $\forall\, U \subseteq Y$ open affine,

$\forall\, U' \subseteq f^{-1}(U) \subseteq X$ open affine,

$\mathcal{O}_X(U')$ is a flat $\mathcal{O}_Y(U)$-module.

Relative dim. n:

$\forall\, V \subseteq Y$ closed irred. subvariety,

$f^{-1}(V) = X \times_Y V$ has pure dim.

$= \dim(V) + n$.

Sheaf of relative differentials $\Omega_{X/Y}$:

$$f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Def $f: X \to X$ is smooth if f is flat of rel. dim. n, some n, and $\Omega_{X/Y}$ locally free $\mathcal{O}_X$-module of rank n.

Properties

1) Flatness and smoothness are preserved under <u>composition</u> and <u>base extension</u>.

2) X alg. scheme / $K = \bar{K}$.

   $X \longrightarrow \mathrm{Spec}(K)$ is smooth $\Leftrightarrow$

   X is non-singular (disjoint union of non-singular varieties of same dimension.)

3) (Grothendieck's generic freeness lemma)

   $f : X \longrightarrow Y$ morphism,

   X alg. scheme, Y irred. variety.

   $\exists$ dense open $U \subseteq Y$:

   $f^{-1}(U) \to U$ flat morphism.

4) $f : X \longrightarrow Y$ morphism of irred. vars.

   Assume: $K = \bar{K}$, $\mathrm{Char}(K) = 0$,

   X non-singular.

   $\exists$ dense open $U \subseteq Y$:

   $f^{-1}(U) \longrightarrow U$ smooth.

Example

$f : X \to Y$ smooth morphism of alg. schemes / $k = \bar{k}$.

$y \in Y$ closed point.

Then $f^{-1}(y)$ is non-singular.

$$\begin{array}{ccc} f^{-1}(y) & \longrightarrow & \{y\} = \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Example

$f : X \to Y$ locally trivial fibration with fiber F:

$Y = \{U_\alpha\}$ open cover, $f^{-1}(U_\alpha) \cong U_\alpha \times F$

$$\begin{array}{ccc} f^{-1}(U_\alpha) & \cong & U_\alpha \times F \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_\alpha & \end{array}$$

Then f is flat.

$k = \bar{k}$: f smooth $\Leftrightarrow$ F non-singular.

<u>Example</u> G alg. group $/ k = \overline{k}$.

$f: X \longrightarrow Y$ G-equivariant morphism of irred. G-varieties.

Assume G acts transitively on Y.
Then f is flat.

Assume $\mathrm{Char}(k) = 0$, X non-singular.

Then f is smooth.

<u>Proof</u> $\exists\, \emptyset \neq U \subseteq Y$ open s.t. $f^{-1}(U) \to U$ is flat/smooth. Then $f^{-1}(s.U) \to s.U$ is also flat/smooth $\forall\, s \in G$.
$\square$

<u>Exer</u> $f: Y \longrightarrow X$, $g: Z \longrightarrow X$ morphisms of irred. varieties.
Then all irred. components of $Y \times_X Z$ have dimension <u>at least</u>

$$\dim(Y) + \dim(Z) - \dim(X).$$

Kleiman's Theorem

G irred. algebraic group $/K=\overline{K}$.
X transitive G-variety.
$f: Y \to X$, $g: Z \longrightarrow X$ morphisms of irred. varieties.

For $s \in G$, let $s.Y$ denote Y with morphism $Y \longrightarrow X$, $y \mapsto s.f(y)$.

(a) $\exists$ dense open $U \subseteq G$:
$\forall s \in U$: $(s.Y) \times_X Z$ has pure dim.:
$$\dim (s.Y) \times_X Z = \dim Y + \dim Z - \dim X.$$

(b) Assume $\mathrm{char}(K) = 0$, Y, Z non-singular. Then:
$\exists$ dense open $U \subseteq G$:
$\forall s \in U$: $(s.Y) \times_X Z$ is a (disjoint union of) non-sing. varieties.

<u>Remark</u>: In (b), can replace "non-singular" with Cohen-Macaulay / normal / reduced / locally irreducible. (Last two already assumed.)

Proof $m : G \times Y \longrightarrow X, \ (g, y) \longmapsto g.F(y)$.

$$\begin{tikzcd}
s.Y \times_X Z \arrow[r] \arrow[d] & (U \times Y) \times_X Z \arrow[r] \arrow[d] & (G \times Y) \times_X Z \arrow[r, "m'"] \arrow[d] & Z \arrow[d, "g"] \\
s.Y \arrow[r] \arrow[d] & U \times Y \arrow[r] \arrow[d] & G \times Y \arrow[r, "m"] \arrow[d, "pr_1"] & X \\
\{s\} \arrow[r] & U \arrow[r] & G &
\end{tikzcd}$$

$m : G \times Y \longrightarrow X$ G-equiv., $G \circlearrowright X$ transitive

$\Rightarrow$ m flat/smooth of rel. dim $=$

$n = \dim(G \times Y) - \dim(X)$.

$\Rightarrow$ m' flat/smooth of rel. dim. n.

$\Rightarrow$ $(G \times Y) \times_X Z$ has pure dim $= n + \dim(Z)$.

/ non-singular of dim. $n + \dim(Z)$.

$\exists\, \emptyset \neq U \subseteq G$ open s.t.

$(U \times Y) \times_X Z \longrightarrow U$ flat/smooth.

of rel. dim. $N = \dim(Y) + \dim(Z) - \dim(X)$.

$s.Y \times_X Z$ of pure dim. N / non-singular, $\forall s \in U$.

$\square$