Determinantal varieties

$H = \mathrm{Mat}(e \times f, K)$.

$G = GL(e) \times GL(f) \circlearrowright H$.

$$(X, Y).M = XHY^{-1}.$$

orbits: $\Omega_r^\circ = \{M \in H \mid \mathrm{rank}(M) = r\}$

$$0 \le r \le \min(e, f).$$

$$\Omega_r = \overline{\Omega_r^\circ} = \{M \in H \mid \mathrm{rank}(M) \le r\}.$$

Dense open subset of Ω_r:

$$\left\{ \begin{array}{|c|c|} \hline A & B \\ \hline C & CA^{-1}B \\ \hline \end{array} \;\middle|\; \begin{array}{l} A \in GL(r, K),\ B \in \mathrm{Mat}(r \times (f-r)) \\ C \in \mathrm{Mat}((e-r) \times r) \end{array} \right\}$$

$\therefore$ Ω_r rational variety,

$\mathrm{codim}(\Omega_r, H) = (e-r)(f-r)$.

Facts:
- $I(\Omega_r)$ is generated by minors of size $r+1$.
- Ω_r is Cohen-Macaulay.

Thom–Porteous degeneracy locus

X alg. scheme.

E, F vector bundles of ranks e, f.

$\phi : E \longrightarrow F$ vector bundle morphism.

Locus of singularities:

$$\Omega_r(\phi) = \{x \in X \mid \operatorname{rank}(E(x) \xrightarrow{\phi} F(x)) \leq r\}$$
$$= Z(\Lambda^{r+1} E \longrightarrow \Lambda^{r+1} F)$$

closed subscheme of X.

Questions: Is $\Omega_r(\phi) = \emptyset$ possible?

$[\Omega_r(\phi)] = ?$

Goal:

Construct $\mathbf{\Omega}_r(\phi) \in A^{(e-r)(d-r)}(\Omega_r(\phi) \xrightarrow{i} X)$

$$i_* \mathbf{\Omega}_r(\phi) = S_\lambda(F-E) \in A^{|\lambda|}(X)$$

$$\lambda = (e-r)^{f-r} = (e-r, \dots, e-r)$$

X CM, $\operatorname{codim}(\Omega_r(\phi), X) = (e-r)(f-r)$

$$\Rightarrow \mathbf{\Omega}_r(\phi) \cap [X] = [\Omega_r(\phi)] \in A_*(\Omega_r(\phi))$$

Zero locus of section

X alg. scheme.

$E \to X$ vector bundle of rank e.

$s : X \to E$ global section.

$$Z(s) = \{x \in X \mid s(x) = 0 \in E(x)\}$$

Local equations

$$E|_U = U \times K^e$$

$$s(x) = (x, s_1(x), \dots, s_e(x)), \quad s_1, \dots, s_e \in \mathcal{O}_X(U)$$

$$Z(s) \cap U = V(s_1, \dots, s_e) \subseteq X \text{ closed subscheme.}$$

Def $s \in \Gamma(X, E)$ is a regular section if $(s_1, \dots, s_e)$ regular sequence in $\mathcal{O}_{p,X}$ $\forall\, p \in Z(s)$.

$\Rightarrow$ $i : Z(s) \to X$ regular embedding with normal bundle $N_{Z(s)}X = E|_{Z(s)}$.

$$\begin{tikzcd}
N_{Z(s)}X = i^*E \arrow[r] \arrow[d] & Z(s) \arrow[r, "i"] \arrow[d, "i"] & X \arrow[d, "s"] \\
N_X E = E \arrow[r] & X \arrow[r, "s_E"] & E
\end{tikzcd}$$

s_E zero section.

Example X CM, $\operatorname{codim}(Z(s), X) = e \Rightarrow$ s regular section.

Localized top Chern class

Assume X has pure dimension.

$s : X \longrightarrow E$ any section.

Then s regular embedding of codim e.

Orientation: $[s] \in A^e(X \xrightarrow{s} E)$:

$[s] \cap [V] = s^![V] \in A_*(X \times_E V)$

$$\begin{tikzcd} X \times_E V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "s"'] & E \end{tikzcd}$$

Def $\mathbb{Z}(s) = s^*[s_E] = s_E^*[s] \in A^e(Z(s) \to X)$

$$\begin{tikzcd} Z(s) \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "s"] \\ X \arrow[r, "s_E"'] & E \end{tikzcd}$$

Properties

(1) $s \in \Gamma(X, E)$ regular section $\Rightarrow$

$\mathbb{Z}(s) \cap [X] = s_E^![X] = i^*[X] = [Z(s)]$.

(2) Each comp. of $Z(s)$ has codim $\geq e$ in X.

$\operatorname{codim}(Z(s), X) = e \Rightarrow$

$\mathbb{Z}(s) \cap [X] \in A_*(Z(s))$ positive cycle with support $Z(s)$.

(3) Commutes with pullbacks:

$f : X' \longrightarrow X$ morphism $\Rightarrow$

$$f^* \mathbb{Z}(s) = \mathbb{Z}(f^* s) \in A^e(Z(f^*s) \xrightarrow{i'} X').$$

<u>Proof</u> $E' = f^*E, \quad s' = f^*s \in \Gamma(X', E')$.

$$\begin{aligned} f^* \mathbb{Z}(s) &= f^* s^* [s_E] \\ &= s'^* f'^* [s_E] \\ &= s'^* [s_{E'}] \\ &= \mathbb{Z}(s'). \end{aligned}$$

$$\begin{tikzcd} Z(s') \arrow[rr, "i'"] \arrow[dr] \arrow[dd, "i'"'] & & X' \arrow[dr, "f"] \arrow[dd, "s'"] & \\ & Z(s) \arrow[rr] \arrow[dd, "i"] & & X \arrow[dd, "s"] \\ X' \arrow[rr] \arrow[dr, "f"'] & & E' \arrow[dr, "f'"] & \\ & X \arrow[rr, "s_E"'] & & E \end{tikzcd}$$

$\square$

(4) $i_* \mathbb{Z}(s) = c_e(E) \in A^e(X)$.

Enough: $i_* \mathbb{Z}(s) \cap [X] = c_e(E) \in A_*(X)$.

$$\begin{aligned} i_* \mathbb{Z}(s) \cap [X] &= i_* s_E^! [X] \\ &= s_E^* s_* [X] \\ &= s^* s_* [X] \\ &= c_e(E) \cap [X]. \end{aligned}$$

$$\begin{tikzcd} Z(s) \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "s"] \\ X \arrow[r, "s_E"'] & E \end{tikzcd}$$

(5) commutes with proper pushforward

$f : X' \longrightarrow X$ proper,

X, X' irred. varieties.

$$\begin{tikzcd} X' \arrow[r, "f"] & X \\ Z(s') \arrow[u] \arrow[r, "g"'] & Z(s) \arrow[u] \end{tikzcd}$$

Then:

$$g_*\left(\mathbb{Z}(s') \cap [X']\right) = \deg(X'/X) \cdot \mathbb{Z}(s) \cap [X]$$
$$\in A_*(Z(s')).$$

$$\begin{tikzcd} Z(s') \arrow[rr, "i'"] \arrow[dr, "g"] \arrow[dd, "i'"'] & & X' \arrow[dr, "f"] \arrow[dd, "s'"] & \\ & Z(s) \arrow[rr] \arrow[dd, "i"] & & X \arrow[dd, "s"] \\ X' \arrow[rr] \arrow[dr, "f"'] & & E' \arrow[dr, "f'"] & \\ & X \arrow[rr, "s_E"'] & & E \end{tikzcd}$$

$$g_* s_{E'}^! [X'] = g_* s_E^! [X']$$
$$= s_E^! f_* [X']$$
$$= \deg(X'/X)\, s_E^! [X]$$

Thom-Porteous class

$\phi : E \longrightarrow F$ morphism of vector bundles of ranks e, f.

$\phi \in \Gamma(X, E^\vee \otimes F)$.

$Z(E \longrightarrow F) = Z(\phi) \subseteq X$ closed subscheme.

$\mathbb{Z}(E \to F) = \mathbb{Z}(\phi) \in A^{ef}(Z(\phi) \longrightarrow X)$.

$0 \leq r \leq \min(e, f)$.

$\Omega_r(\phi) = Z(\Lambda^{r+1}\phi) \subseteq X$.

Expected codim $= (e-r)(f-r)$.

<u>want</u>: $\mathbf{\Omega}_r(\phi) \in A^{(e-r)(f-r)}(\Omega_r(\phi) \to X)$.

$\mathbf{\Omega}_0(\phi) = \mathbb{Z}(\phi)$.

<u>Note</u>: can't use $\mathbb{Z}(\Lambda^{r+1}\phi)$ when $r > 0$ since rank of $\Lambda^{r+1}E^\vee \otimes \Lambda^{r+1}F$ is too large.

$G = Gr(r, F) \xrightarrow{\pi} X$ Grassmann bundle.

On G: $\quad 0 \to R \to F \to F/R \to 0$

$$\begin{tikzcd} Z(E \to F/R) \arrow[r, hook, "j"] \arrow[d, "\pi'"'] & G \arrow[d, "\pi"] \\ \Omega_r(E \to F) \arrow[r, hook, "i"] & X \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, dashed] & F \arrow[r] & F/R \\ & R \arrow[u] & \end{tikzcd}$$

$\mathbb{Z}(E \to F/R) \in A(Z(E \to F/R) \to G)$

$[\pi] \in A(G \xrightarrow{\pi} X)$.

<u>Def</u> $\Omega_r(E \xrightarrow{\phi} F) = \pi'_*(\mathbb{Z}(E \to F/R) \cdot [\pi])$

$$\in A^{(e-r)(f-r)}(\Omega_r(\phi) \xrightarrow{i} X)$$

<u>Compute</u>:

$$\begin{aligned} i_* \Omega_r(\phi) &= \pi_* j_* (\mathbb{Z}(E \to F/R) \cdot [\pi]) \\ &= \pi_* (c_{e(f-r)}(E^\vee \otimes F/R) \cdot [\pi]) \\ &= \pi_* (S_{(e)^{f-r}}(F/R - E) \cdot [\pi]) \\ &= S_\lambda(F - E) \in A^{|\lambda|}(X) \end{aligned}$$

$$\lambda = (e-r)^{f-r} = (e-r, \ldots, e-r)$$