Thom-Porteous formula

X alg. scheme.

$\phi : E \longrightarrow F$ morphism of vector bundles of ranks e, f.

$0 \le r \le \min(e,f)$.

$$\Omega_r(\phi) = \{x \in X \mid \operatorname{rank}(E(x) \xrightarrow{\phi} F(x)) \le r\}$$
$$= Z(\Lambda^{r+1}\phi)$$

$$G = Gr(r, F) \xrightarrow{\pi} X$$
$$0 \longrightarrow R \longrightarrow F \longrightarrow F/R \longrightarrow 0$$

$$\begin{tikzcd} Z(E \to F/R) \arrow[r] \arrow[d, "\bar\pi"'] & G \arrow[d, "\pi"] \\ \Omega_r(\phi) \arrow[r, "i"] & X \end{tikzcd}$$

$$\mathbb{Z}(E \longrightarrow F/R) \in A^{e(f-r)}(Z(E \to F/R) \to G)$$

$$\Omega_r(\phi) = \bar\pi_*(\mathbb{Z}(E \to F/R) \cdot [\pi])$$
$$\in A^{(e-r)(f-r)}(\Omega_r(\phi) \xrightarrow{i} X)$$

Properties

(1) $i_* \Omega_r(\phi) = S_\lambda(F - E) \in A^{(e-r)(f-r)}(X)$

$\lambda = (e-r)^{f-r}$.

(2) Commutes with pullbacks:

$f : X' \longrightarrow X$ morphism $\Rightarrow$

$\Omega_r(f^*\phi) = f^*\Omega_r(\phi)$

Proof

$E' = f^*E, \quad F' = f^*F, \quad \phi' = f^*\phi$

$\Omega = \Omega_r(\phi), \quad \Omega' = \Omega_r(\phi')$

$$\begin{tikzcd} E' \arrow[r, "\phi'"] \arrow[dr] & F' \arrow[d] & E \arrow[r, "\phi"] \arrow[d] & F \arrow[dl] \\ & X' \arrow[rr] & X & \\ & \Omega' \arrow[u] \arrow[rr] & \Omega \arrow[u] & \end{tikzcd}$$

$G = Gr(r, F), \quad Z = Z(E \longrightarrow F/R) \subseteq G$

$G' = Gr(r, F'), \quad Z' = Z(E' \longrightarrow F'/R') \subseteq G'$

$$
\begin{tikzcd}
Z' \arrow[rr] \arrow[dr] \arrow[dd, "\bar{\pi}'"'] & & G' \arrow[dr, "h"] \arrow[dd, "\pi'" near start] & \\
& Z \arrow[rr, crossing over] & & G \arrow[dd, "\pi"] \\
\Omega' \arrow[rr] \arrow[dr] & & X' \arrow[dr, "f"] & \\
& \Omega \arrow[rr] \arrow[from=uu, crossing over, "\bar{\pi}" near start] & & X
\end{tikzcd}
$$

$$
\begin{aligned}
f^* \Omega_r(\phi) &= f^* \bar{\pi}_* \left(\mathbb{Z}(E \to F/R) \cdot [\pi] \right) \\
&= \bar{\pi}'_* f^* \left(\mathbb{Z}(E \to F/R) \cdot [\pi] \right) \\
&= \bar{\pi}'_* \left(h^* \mathbb{Z}(E \to F/R) \cdot f^* [\pi] \right) \\
&= \bar{\pi}'_* \left(\mathbb{Z}(E' \to F'/R') \cdot [\pi'] \right) \\
&= \Omega_r(\phi').
\end{aligned}
$$

□

Tautological sections

$\pi : E \longrightarrow X$ vector bundle.

$$\begin{tikzcd} \pi^*E \arrow[r, "p_2"'] \arrow[d, "p_1"] & E \arrow[d] \\ E \arrow[r, "\pi"'] \arrow[u, bend left, "\tilde{\sigma}"] & X \end{tikzcd}$$

$p_1 \tilde{\sigma} = p_2 \tilde{\sigma} = id_E .$

$\tilde{\sigma} \in \Gamma(E, \pi^*E)$ tautological section.

Universal property

$$\mathrm{Mor}(Z, E) = \left\{ (f, \sigma) : \begin{array}{l} f : Z \longrightarrow X \text{ morphism} \\ \sigma \in \Gamma(Z, f^*E) \end{array} \right\}$$

$$h \longmapsto (\pi \circ h , \ h^* \tilde{\sigma})$$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ Z \arrow[r, "h"'] \arrow[u, bend left, "\sigma = h^*\tilde{\sigma}"] \arrow[rr, bend right, "f"'] & E \arrow[r, "\pi"'] \arrow[u, bend left, "\tilde{\sigma}"] & X \end{tikzcd}$$

Tautological morphism

E, F vector bundles on X.

$$H = \mathrm{Hom}(E,F) = E^\vee \otimes F \xrightarrow{\pi} X$$

Taut. section:

$$\tilde\phi \in \Gamma(H, (\pi^*E)^\vee \otimes \pi^*F)$$

$$\tilde\phi : \pi^*E \longrightarrow \pi^*F$$

universal morphism on H.

$$\mathrm{Mor}(Z,H) = \left\{ (f,\phi) : \begin{array}{l} f : Z \longrightarrow X \\ \phi : f^*E \longrightarrow f^*F \end{array} \right\}$$

$$h \longmapsto (\pi \circ h, h^*\tilde\phi)$$

$$\begin{tikzcd}
f^*E \arrow[r, "h^*\tilde\phi"] \arrow[d] & f^*F & \pi^*E \arrow[r, "\tilde\phi"] \arrow[d] & \pi F & E \arrow[d] & F \arrow[dl] \\
Z \arrow[rr, "h"'] \arrow[rrrr, bend right, "f"'] & & H \arrow[rr, "\pi"'] & & X &
\end{tikzcd}$$

Universal degeneracy locus

$H = \mathrm{Hom}(E, F) \xrightarrow{\pi} X.$

$\tilde{\phi} : \pi^* E \longrightarrow \pi^* F$ tautological morphism on H.

$\tilde{\Omega}_r = \Omega_r(\tilde{\phi}) \subseteq H$

Note: If $U \subseteq X$ open, $E|_U = U \times K^e$, $F|_U = U \times K^f$.

$\pi^{-1}(U) = U \times \mathrm{Mat}(f \times e).$

$\pi^{-1}(U) \cap \tilde{\Omega}_r = U \times \Omega_r$

$\mathrm{codim}(\tilde{\Omega}_r, H) = (e-r)(f-r).$

X CM $\Rightarrow$ $\tilde{\Omega}_r$ CM.

Given $\phi : E \longrightarrow F$ on X:

$\exists!\ f : X \longrightarrow H, \quad \phi = f^*(\tilde{\phi}).$

$\Omega_r(\phi) = f^{-1}(\tilde{\Omega}_r)$

Cor

Each irreducible component of $\Omega_r(\phi)$ has codim. $\leq (e-r)(f-r)$ in X.

If X is Cohen-Macaulay and $\operatorname{codim}(\Omega_r(\phi), X) = (e-r)(f-r)$ then $\Omega_r(\phi)$ is Cohen-Macaulay.

Proof

$$\begin{tikzcd} X \arrow[r, "f"] & H \\ \Omega_r(\phi) \arrow[u] \arrow[r] & \tilde{\Omega}_r \arrow[u] \end{tikzcd}$$

$\Omega_r(\phi) \subseteq \tilde{\Omega}_r$ is locally defined by $\operatorname{rank}(H) = ef$ equations.

□

$H = \mathrm{Hom}(E, F) \longrightarrow X$

$G = \mathrm{Gr}(r, F) \longrightarrow X$

$G \times_X H$ = Grassmann bundle over H

= Hom bundle over G.

universal data: $\begin{tikzcd} E \arrow[r, "\tilde{\phi}"] & F \arrow[r] & F/R \\ R \arrow[u] & & \end{tikzcd}$

$\tilde{Z} = Z(E \xrightarrow{\tilde{\phi}} F/R) \subseteq G \times_X H$

$\tilde{\Omega}_r = \Omega_r(\tilde{\phi}) \subseteq H$

<u>Claim</u>: $p = pr_H : \tilde{Z} \longrightarrow \tilde{\Omega}_r$ is "birational."

dense open $\xrightarrow{\cong}$ dense open.

<u>Note</u>: $\tilde{Z} \cong \mathrm{Hom}(E, R) \longrightarrow G$

universal data: $E \to R \subseteq F$.

$\tilde{Z}^\circ \subseteq \tilde{Z}$ locus where $E \twoheadrightarrow R$.

universal data: $E \twoheadrightarrow R \subseteq F$

$$\tilde{\Omega}_r^\circ = \{ h \in H \mid \operatorname{rank}(E(h) \xrightarrow{\tilde{\phi}} F(h)) = r \} \subseteq \tilde{\Omega}_r$$

universal data: $E \twoheadrightarrow \tilde{\phi}(E) \subseteq F$

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{p} & \tilde{\Omega}_r \\ \cup\!| & & \cup\!| \\ \tilde{Z}^\circ & \xrightarrow{\tilde{\Xi}} & \tilde{\Omega}_r^\circ \end{array}$$

<u>Note</u>: X non-singular $\Rightarrow$

$p : \tilde{Z} \longrightarrow \tilde{\Omega}_r$ resolution of singularities!

$$\begin{tikzcd} G \times_X H \arrow[r, "\pi"] & H \\ \tilde{Z} \arrow[u] \arrow[r, "\sim", "p"'] & \tilde{\Omega}_r \arrow[u] \end{tikzcd}$$

$$\begin{aligned}
[\tilde{\Omega}_r] &= p_*[\tilde{Z}] \\
&= p_*\left(\mathbb{Z}(E \xrightarrow{\tilde{\phi}} F/R) \cap [G \times_X H]\right) \\
&= p_*\left(\mathbb{Z}(E \to F/R) \cap \pi^*[H]\right) \\
&= \pi_*\left(\mathbb{Z}(E \to F/R) \cdot [\pi]\right) \cap [H] \\
&= \Omega_r(\tilde{\phi}) \cap [H].
\end{aligned}$$

$\phi : E \to F$ on X

$f : X \to H, \quad \phi = f^* \tilde{\phi}$

$$\begin{tikzcd} \Omega_r(\phi) \arrow[r, "i"] \arrow[d] & \tilde{\Omega}_r \arrow[d] \\ X \arrow[r, "f"] & H \end{tikzcd}$$

Compatible with pullback $\Rightarrow$

$$\Omega_r(\phi) \cap [X] = f^!\big(\Omega_r(\tilde{\phi}) \cap [H]\big)$$
$$= f^!\big([\tilde{\Omega}_r]\big)$$

<u>Properties:</u>

(3) $\operatorname{codim}(\Omega_r(\phi), X) = (e-r)(f-r) \Rightarrow$
$\Omega_r(\phi) \cap [X]$ positive cycle with support $\Omega_r(\phi)$.

(4) X CM and $\operatorname{codim}(\Omega_r(\phi), X) = (e-r)(f-r)$
$\Rightarrow [\Omega_r(\phi)] = \Omega_r(\phi) \cap [X]$

(5) $\Omega_r(-)$ commutes with pushforward.

Cor X CM variety,

$$\operatorname{codim}(\Omega_r(\phi), X) = (e-r)(f-r)$$

$$\Rightarrow [\Omega_r(\phi)] = S_\lambda(F-E) \cap [X] \in A_*(X)$$

Localized Chern/Segre classes

E vector bundle on X of rank e.

$\phi : X \times K^q \longrightarrow E$ any morphism from trivial bundle.

$$\Omega_r(\phi) \xrightarrow{i} X$$

$$i_* \Omega_r(\phi) = S_\lambda(E) \in A^*(X), \quad \lambda = (q-r)^{e-r}$$

Assume $q = e-p+1$, $r = e-p$:

$$i_* \Omega_{e-p}(\phi) = S_{(1)^p}(E) = c_p(E)$$

$$\Omega_{e-p}(\phi) \in A^p(\Omega_{e-p}(\phi) \to X)$$

localized Chern class.

Assume $q = e+p-1$, $r = e-1$:

$$i_* \Omega_{e-1}(\phi) = S_p(E)$$

$$\Omega_{e-1}(\phi) \in A^p(\Omega_{e-1}(\phi) \to X)$$

localized Segre class.