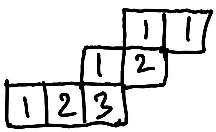
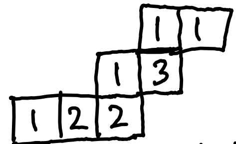
Littlewood - Richardson rule

$T \in SSYT(\nu/\lambda)$.

T is a <u>LR tableau</u> if each occurrence of $x \geq 2$ in the row word is followed by more occurrences of $x-1$ than x.



LR tableau

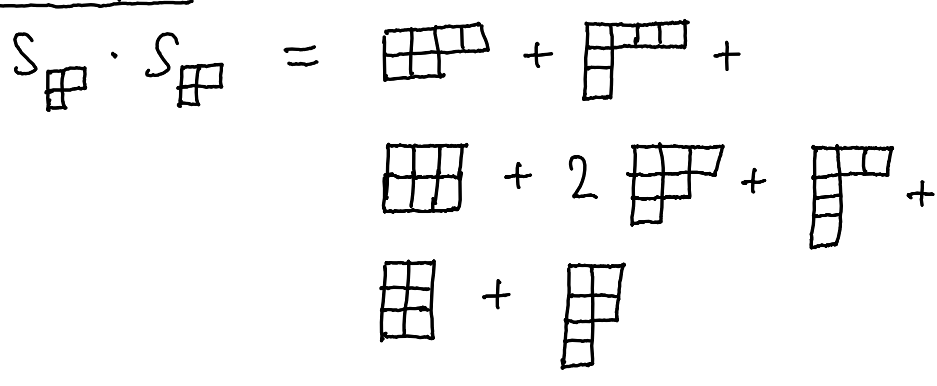


Not LR tableau.

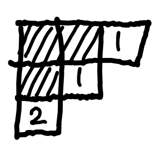
Theorem

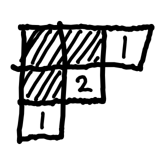
$$c^{\nu}_{\lambda\mu} = \# \text{ LR tableaux of shape } \nu/\lambda \text{ with content } \mu.$$

Example



Coef. of  :





Exercise

LR tableaux are preserved by jdt slides.

$$
\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & \square & 1 & 2 & 2 & \\
1 & 2 & 2 & 3 & & \\
3 & 3 & 4 & & &
\end{array}
\longrightarrow
\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & 1 & \square & 2 & 2 & \\
1 & 2 & 2 & 3 & & \\
3 & 3 & 4 & & &
\end{array}
\longrightarrow
\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & 1 & 2 & 2 & 2 & \\
1 & 2 & \square & 3 & & \\
3 & 3 & 4 & & &
\end{array}
$$

$$
\downarrow
$$

$$
\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & 1 & 2 & 2 & 2 & \\
1 & 2 & 3 & \square & & \\
3 & 3 & 4 & & &
\end{array}
$$

Exercise

R associative ring with 1.

$S \subseteq R$ set of generators as $\mathbb{Z}$-algebra.

M left R-module.

$\phi : R \times M \longrightarrow M$ $\mathbb{Z}$-bilinear map.

Assume: $\forall\, s \in S,\ r \in R,\ m \in M$:

$$\phi(1, m) = m$$

$$\phi(sr, m) = s\,\phi(r, m)$$

Then $\phi(r, m) = rm \quad \forall\, (r, m) \in R \times M.$

Def $\hat{c}^{\nu}_{\lambda\mu}$ = # LR tableaux of shape ν/λ with content μ.

Def $\phi : \Lambda \times \Lambda \longrightarrow \Lambda$

$$\phi(s_\lambda, s_\mu) = \sum_{\nu} \hat{c}^{\nu}_{\lambda\mu} s_\nu$$

To prove LR rule, enough to show

(1) $\phi(1, s_\mu) = s_\mu$

(2) $\phi(s_p s_\lambda, s_\mu) = s_p \, \phi(s_\lambda, s_\mu)$

This implies $\phi(s_\lambda, s_\mu) = s_\lambda \cdot s_\mu$

$$\Rightarrow c^{\nu}_{\lambda\mu} = \hat{c}^{\nu}_{\lambda\mu}.$$

(1) $\exists!$ LR-tableau of shape $\nu = \nu/0$:

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2&2\\\cline{1-4} 3&3\\\cline{1-2}\end{array}$$

content $= \nu$.

(2) $\phi(S_p S_{\lambda'}, S_\mu) = S_p\, \phi(S_{\lambda'}, S_\mu)$:

$$\text{LHS} = \sum_{\substack{\lambda:\ \lambda/\lambda' \text{ horiz} \\ |\lambda/\lambda'| = p}} \phi(S_\lambda, S_\mu)$$

$$= \sum_{\substack{\lambda, \nu:\ \lambda/\lambda' \text{ horiz} \\ |\lambda/\lambda'| = p}} \hat{C}^{\nu}_{\lambda\mu}\, S_\nu$$

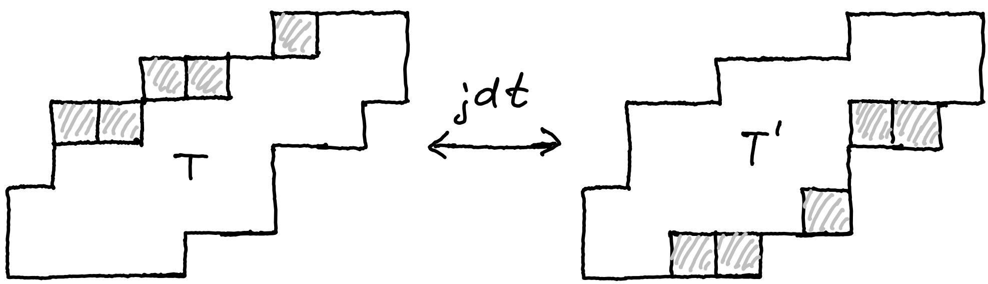
$$\text{RHS} = S_p \cdot \sum_{\nu'} C^{\nu'}_{\lambda'\mu}\, S_{\nu'}$$

$$= \sum_{\substack{\nu', \nu:\ \nu/\nu' \text{ horiz} \\ |\nu/\nu'| = p}} C^{\nu'}_{\lambda'\mu}\, S_\nu$$

Show: Given p, λ', μ, ν:

$$\sum_{\substack{\lambda:\ \lambda/\lambda' \text{ horiz} \\ |\lambda/\lambda'| = p}} \hat{C}^{\nu}_{\lambda\mu} = \sum_{\substack{\nu':\ \nu/\nu' \text{ horiz} \\ |\nu/\nu'| = p}} \hat{C}^{\nu'}_{\lambda'\mu'}$$

$$\coprod_{\lambda} \mathrm{LRTAB}(\nu/\lambda\,;\mu) \longleftrightarrow \coprod_{\nu'} \mathrm{LRTAB}(\nu'/\lambda'\,;\mu)$$



<u>Recall:</u> $\lambda, \nu \in \mathbb{Z}^\ell$ partitions:

$$S_{\nu/\lambda} = \det\left(S_{\nu_i - i - \lambda_j + j}\right)_{\ell\times\ell} \in \Lambda.$$

$$S_{\nu/\lambda} \neq 0 \Rightarrow \lambda \subseteq \nu.$$

<u>Thm</u> $S_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda\mu} S_\mu.$

<u>Example</u> $S_{(3,2)/(1)} = S_{(2,2)} + S_{(3,1)}$ $\quad \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline 2 & 2 & \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline 1 & 2 & \\ \hline\end{array}$

<u>Example</u> $S_\nu(x;z) = \sum_\lambda S_\lambda(x;y)\, S_{\nu/\lambda}(y;z)$

$$= \sum_{\lambda,\mu} c^\nu_{\lambda\mu} S_\lambda(x;y)\, S_\mu(y;z)$$

Def. $d^\nu_{\lambda\mu} \in \mathbb{Z}$ by $S_{\nu/\lambda} = \sum_\mu d^\nu_{\lambda\mu} S_\mu.$

Def. $\phi : \Lambda \times \Lambda \longrightarrow \Lambda$

$$\phi(S_\lambda, S_\mu) = \sum_\nu d^\nu_{\lambda\mu} S_\nu.$$

<u>Show:</u> $\phi(S_\lambda, S_\mu) = S_\lambda \cdot S_\mu$

(1) $\phi(1, S_\mu) = S_\mu$:

$$d^\nu_{0\mu} = \delta_{\mu,\nu} \Leftrightarrow S_{\nu/0} = S_\nu.$$

(2) $\phi(S_p S_{\lambda'}, S_\mu) = S_p \phi(S_{\lambda'}, S_\mu)$

<u>Show:</u> Given p, λ', μ, ν:

$$\sum_{\substack{\lambda : \lambda/\lambda' \text{ horiz} \\ |\lambda/\lambda'| = p}} d^{\nu}_{\lambda\mu} = \sum_{\substack{\nu' : \nu/\nu' \text{ horiz} \\ |\nu/\nu'| = p}} d^{\nu'}_{\lambda'\mu}$$

Let $x = (x_1, x_2, \ldots, x_N)$, z single variable.

$$S_{\nu/\lambda'}(z, x) = \sum_{\lambda} S_{\lambda/\lambda'}(z) S_{\nu/\lambda}(x)$$

$$= \sum_{\lambda, \mu : \lambda/\lambda' \text{ horiz}} z^{|\lambda/\lambda'|} d^{\nu}_{\lambda\mu} S_\mu(x)$$

$$S_{\nu/\lambda'}(x, z) = \sum_{\nu'} S_{\nu'/\lambda'}(x) S_{\nu/\nu'}(z)$$

$$= \sum_{\nu', \mu : \nu/\nu' \text{ horiz}} d^{\nu'}_{\lambda'\mu} S_\mu(x) z^{|\nu/\nu'|}$$

Extract coef. of $z^p S_\mu(x)$.

Bialgebra / K:

K-algebra H with

<u>unit</u> $\eta: K \longrightarrow H, \quad 1 \mapsto 1$

<u>product</u> $\nabla: H \otimes H \longrightarrow H, \quad a \otimes b \mapsto ab$

<u>counit</u> $\varepsilon: H \longrightarrow K$

<u>coproduct</u> $\Delta: H \longrightarrow H \otimes H$

such that

- ε and Δ are K-alg. homs.
- Δ is <u>coassociative</u>:
  $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$:
  $$H \longrightarrow H \otimes H \longrightarrow H \otimes H \otimes H$$
- ε is a <u>counit</u>:
  $(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta$

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & H \otimes H \arrow[d, "id \otimes \varepsilon"] \\ H \otimes H \arrow[r, "\varepsilon \otimes id"'] & H \end{tikzcd}$$

Hopf algebra $/K$

Bialgebra H with

<u>antipode</u> $S: H \longrightarrow H$

such that diagram commutes:

$$\begin{tikzcd}
 & H\otimes H \arrow[r, "S\otimes id"] & H\otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & K \arrow[rr, "\eta"] & & H \\
 & H\otimes H \arrow[r, "id\otimes S"'] & H\otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

<u>involutive</u>: $S^2 = id$

Example

$\mathrm{Spec}(H)$ linear alg. group $/K = \overline{K}$

$\Leftrightarrow$ H f.g. involutive Hopf alg. $/K$.

Λ Hopf algebra $/\mathbb{Z}$

unit: $\eta : \mathbb{Z} \longrightarrow \Lambda \, , \quad 1 \mapsto 1$

product: $\nabla : \Lambda \otimes \Lambda \longrightarrow \Lambda$

$$S_\lambda \otimes S_\mu \longmapsto \sum c^{\nu}_{\lambda\mu} S_\nu$$

counit: $\varepsilon : \Lambda \longrightarrow \Lambda$

$$S_\lambda \longmapsto S_\lambda(0;0) = \delta_{\lambda,0}$$

coproduct: $\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$

$$S_\nu \longmapsto \sum_{\lambda,\mu} c^{\nu}_{\lambda\mu} S_\lambda \otimes S_\mu$$

Autipode: $S : \Lambda \longrightarrow \Lambda$

$$S_\lambda \longmapsto (-1)^{|\lambda|} S_{\lambda^T} .$$

Δ $\mathbb{Z}$-algebra hom:

$$S_{\nu'}(x,y)\cdot S_{\nu''}(x,y) = \sum_{\nu} c^{\nu}_{\nu'\nu''} S_\nu(x,y)$$

$$\| \qquad = \sum_{\nu,\lambda,\mu} c^{\nu}_{\nu'\nu''} c^{\nu}_{\lambda\mu} S_\lambda(x) S_\mu(y)$$

$$\sum_{\lambda',\mu'} c^{\nu'}_{\lambda'\mu'} S_{\lambda'}(x) S_{\mu'}(y) \sum_{\lambda'',\mu''} c^{\nu''}_{\lambda''\mu''} S_{\lambda''}(x) S_{\mu''}(y)$$

$$= \sum_{\lambda',\mu',\lambda'',\mu'',\lambda,\mu} c^{\nu'}_{\lambda'\mu'} c^{\nu''}_{\lambda''\mu''} c^{\lambda}_{\lambda'\lambda''} c^{\mu}_{\mu'\mu''} S_\lambda(x) S_\mu(y)$$

Equiv. to:

$$\Delta(S_{\nu'}\cdot S_{\nu''}) = \Delta(S_{\nu'})\cdot\Delta(S_{\nu''})$$

S antipode:

$$\delta_{\nu,0} = S_\nu(x\,;x)$$

$$= \sum_{\lambda,\mu} C^{\nu}_{\lambda\mu}\, S_\lambda(x)\, S_\mu(0;x)$$

$$= \sum_{\lambda,\mu} C^{\nu}_{\lambda\mu}\, S_\lambda(x)\, (-1)^{|\mu|} S_{\mu^T}(x)$$

$$= \sum_{\lambda,\mu,\tau} C^{\nu}_{\lambda\mu}\, (-1)^{|\mu|}\, C^{\tau}_{\lambda\mu^T}\, S_\tau(x)$$

Equivalent to:

$$\eta\,\varepsilon(S_\nu) = \nabla(\mathrm{id}\otimes S)\Delta(S_\nu).$$