K-theory

X algebraic scheme $/k$.

K-cohomology:

$K^0(X)$ = Grothendieck group of vector bundles
= free abelian group gen. by vector bdls
$/ [E] = [E'] + [E'']$ if $\exists\ 0 \to E' \to E \to E'' \to 0$

Commutative ring: $[E]\cdot[F] = [E \otimes F]$.

Any morphism $f: Y \longrightarrow X$
defines ring hom.
$f^*: K^0(X) \longrightarrow K^0(Y)$, $[E] \mapsto [f^*E]$.

Functor: $K^0 : Sch_k^{op} \longrightarrow Ring$

Example $K^0(\mathbb{A}^n) = \mathbb{Z}$.

Serre's conjecture: Every vector bundle on $\mathbb{A}^n$ is trivial.

K-homology

$K_0(X)$ = free abelian group gen by coherent $\mathcal{O}_X$-modules /

$[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$

if $\exists\ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

$K^0(X)$-module:

$$[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}]$$

Check: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \Rightarrow$

$$0 \to E \otimes \mathcal{F}' \to E \otimes \mathcal{F} \to E \otimes \mathcal{F}'' \to 0$$

$0 \to E' \to E \to E'' \to 0 \Rightarrow$

$$0 \to E' \otimes \mathcal{F} \to E \otimes \mathcal{F} \to E'' \otimes \mathcal{F} \to 0$$

Flat pullback: $f : Y \to X$ flat morphism

defines $f^* : K_0(X) \to K_0(Y)$.

Proper pushforward

$f : X \longrightarrow Y$ proper morphism.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

$$R^j f_* \mathcal{F} = \left[U \mapsto H^j(f^{-1}(U), \mathcal{F}) \right]^+$$

Fact:

(1) f proper, $\mathcal{F}$ coherent $\Rightarrow R^j f_* \mathcal{F}$ coherent.

(2) $R^j f_* \mathcal{F} = 0$ for $i > \max\{\dim X_y \mid y \in Y\}$.

Def $f_* : K_0(X) \longrightarrow K_0(Y)$

$$f_*[\mathcal{F}] = \sum_{j \geq 0} (-1)^j [R^j f_* \mathcal{F}] \in K_0(Y).$$

Well defined:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

$\Rightarrow$ LES $0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to R^1 f_* \mathcal{F}' \to \cdots$

$f_*[\mathcal{F}'] + f_*[\mathcal{F}''] - f_*[\mathcal{F}]$

$=$ alternating sum of LES

$= 0 \in K_0(Y)$.

Example $f : X \longrightarrow S = \mathrm{Spec}(K)$ proper.

$$f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}) = \chi(\mathcal{F}) \in K_0(S) = \mathbb{Z}$$

Note: Let $\mathcal{F}_\bullet$ be finite complex of coherent $\mathcal{O}_X$-modules:

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_\ell \to 0$$

Set $\mathcal{H}_i = \operatorname{Ker}(\mathcal{F}_i \to \mathcal{F}_{i+1}) / \operatorname{Im}(\mathcal{F}_{i-1} \to \mathcal{F}_i)$

Then $\sum_{i \geq 0} (-1)^i [\mathcal{F}_i] = \sum_{j \geq 0} (-1)^j [\mathcal{H}_j] \in K_0(X)$.

Functoriality:

$X \xrightarrow{f} Y \xrightarrow{g} Z$ proper morphisms.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

Then $g_* f_* [\mathcal{F}] = (gf)_* [\mathcal{F}] \in K_0(Y)$.

Check:

$$\sum_{i,j \geq 0} (-1)^{i+j} [R^i g_* R^j f_* \mathcal{F}] = \sum_{k \geq 0} (-1)^k [R^k (gf)_* \mathcal{F}] \in K_0(Z)$$

Grothendieck spectral sequence:

$$R^i g_* R^j f_* \mathcal{F} \Rightarrow R^{i+j} (gf)_* \mathcal{F}$$

Projection formula

$f : X \longrightarrow Y$ proper.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

E vector bundle on Y.

$$R^i f_*(f^*E \otimes \mathcal{F}) = E \otimes R^i f_* \mathcal{F}$$

$$\Rightarrow f_*(f^*[E] \cdot [\mathcal{F}]) = [E] \cdot f_*[\mathcal{F}] \in K_0(Y)$$

Note: $K_0(X)$ is a $K^0(Y)$-module through $f^* : K^0(Y) \longrightarrow K^0(X)$.

$f_* : K_0(X) \longrightarrow K_0(Y)$ homomorphism of $K^0(Y)$-modules.

Lemma X non-singular variety.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

$\exists\ \mathcal{E} \twoheadrightarrow \mathcal{F}$ surjective, $\mathcal{E}$ locally free of finite rank.

Proof

$\mathcal{F}$ generated by finitely many sections $\tau \in \Gamma(U, \mathcal{F})$ for which $U = X \setminus D$ complement of effective Cartier div. D.

$L = \mathcal{O}_X(D)$ line bundle.

$s \in \Gamma(X, L)$ global section: $Z(s) = D$.

$\exists N > 0$: $s^N \otimes \tau \in \Gamma(U, L^{\otimes N} \otimes \mathcal{F})$

extends to global section

$\tau' \in \Gamma(X, L^{\otimes N} \otimes \mathcal{F})$.

$L^{\otimes -N} \xrightarrow{\otimes \tau'} \mathcal{F}$ global $\mathcal{O}_X$-hom.

$\tau = s^{-N} \otimes \tau'$ image of $s^{-N} \in \Gamma(U, L^{\otimes -N})$.

$\therefore \bigoplus_{\text{finite}} L_i \twoheadrightarrow \mathcal{F}$.

$\square$

Cor X non-singular variety, $\mathcal{F}$ coherent $\mathcal{O}_X$-module.

$\exists$ finite resolution by locally free $\mathcal{O}_X$-modules of finite rank:

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

Proof

Lemma $\Rightarrow$ $\exists$ exact seq.

$$\mathcal{E}_{n-1} \xrightarrow{\varphi_{n-1}} \mathcal{E}_{n-2} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where $n = \dim(X)$.

Set $\mathcal{E}_n = \mathrm{Ker}(\varphi_{n-1})$.

Let $x \in X$.

$\mathcal{O}_{x,X}$ regular local ring of dim $\leq n$

$\Rightarrow \mathcal{O}_{x,X}$ has homological dim. $\leq n$.

$\Rightarrow (\mathcal{E}_n)_x$ free $\mathcal{O}_{x,X}$-module.

$\therefore$ $\mathcal{E}_n$ locally free.

$\square$

X alg. scheme $/k$.

<u>Lemma 1</u>

Let $\varphi_1 : \mathcal{F}_1 \to \mathcal{F}$ and $\varphi_2 : \mathcal{F}_2 \to \mathcal{F}$ be surjective morphisms of coherent $\mathcal{O}_X$-modules. Then $\exists$ coherent $\mathcal{O}_X$-mod. $\mathcal{E}$ with surjective morphisms

$\psi_1 : \mathcal{E} \to \mathcal{F}_1$, $\psi_2 : \mathcal{E} \to \mathcal{F}_2$

such that $\varphi_1 \psi_1 = \varphi_2 \psi_2$:

$$\begin{tikzcd} & \mathcal{F}_1 \arrow[dr, two heads, "\varphi_1"] & \\ \mathcal{E} \arrow[ur, two heads, "\psi_1"] \arrow[dr, two heads, "\psi_2"'] & & \mathcal{F} \\ & \mathcal{F}_2 \arrow[ur, two heads, "\varphi_2"'] & \end{tikzcd}$$

<u>Proof</u>

$\mathcal{E} = \ker(\varphi_1 - \varphi_2 : \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F})$.

$\psi_i : \mathcal{E} \xrightarrow{\subset} \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_i$ projection.

Then $\varphi_1 \psi_1 = \varphi_2 \psi_2$ is clear.

Given $\alpha_1 \in (\mathcal{F}_1)_x$, choose $\alpha_2 \in (\mathcal{F}_2)_x$ such that $\varphi_2(\alpha_2) = \varphi_1(\alpha_1)$. Then $(\alpha_1, \alpha_2) \in \mathcal{E}_x$ and $\psi_1(\alpha_1, \alpha_2) = \alpha_1$.

$\square$

Lemma 2

Let $\varphi_1 : \mathcal{E}_1 \to \mathcal{F}_1$, $\varphi_2 : \mathcal{E}_2 \to \mathcal{F}_2$,

$\mu_1 : \mathcal{F} \to \mathcal{F}_1$, $\mu_2 : \mathcal{F} \to \mathcal{F}_2$

be surjective morphisms of coherent $\mathcal{O}_X$-modules. Then $\exists$ coherent $\mathcal{O}_X$-module $\mathcal{E}$ with surjective morphisms $\eta_1 : \mathcal{E} \to \mathcal{E}_1$, $\eta_2 : \mathcal{E} \to \mathcal{E}_2$, $\varphi : \mathcal{E} \to \mathcal{F}$, such that

$\mu_i \varphi = \varphi_i \eta_i$, $i = 1, 2$.

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, two heads, "\varphi_1"] & \mathcal{F}_1 \\
\mathcal{E} \arrow[u, two heads, "\eta_1"'] \arrow[r, two heads, "\varphi"'] \arrow[d, two heads, "\eta_2"] & \mathcal{F} \arrow[u, two heads, "\mu_1"'] \arrow[d, two heads, "\mu_2"] \\
\mathcal{E}_2 \arrow[r, two heads, "\varphi_2"'] & \mathcal{F}_2
\end{tikzcd}$$

Proof

Apply Lemma 2 3 times:

$$\begin{tikzcd}
 & & \mathcal{E}_1 \arrow[r, two heads, "\varphi_1"] & \mathcal{F}_1 \\
 & \mathcal{E}_1' \arrow[ur, two heads] \arrow[drr, two heads] & & \\
\mathcal{E} \arrow[ur, two heads] \arrow[dr, two heads] & & & \mathcal{F} \arrow[uu, two heads, "\mu_1"'] \arrow[dd, two heads, "\mu_2"] \\
 & \mathcal{E}_2' \arrow[urr, two heads] \arrow[dr, two heads] & & \\
 & & \mathcal{E}_2 \arrow[r, two heads, "\varphi_2"'] & \mathcal{F}_2
\end{tikzcd}$$

$\square$

Lemma 3

X non-singular variety.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

$\mathcal{E}'_\bullet \to \mathcal{F} \to 0$ and $\mathcal{E}''_\bullet \to \mathcal{F} \to 0$

finite locally free resolutions of $\mathcal{F}$.

Then $\exists$ finite locally free resolution

$\mathcal{E}_\bullet \to \mathcal{F} \to 0$ and surjective morphisms

of complexes $\varphi' : \mathcal{E}_\bullet \to \mathcal{E}'_\bullet$, $\varphi'' : \mathcal{E}_\bullet \to \mathcal{E}''_\bullet$.

such that the diagram commutes:

$$\begin{tikzcd}
\mathcal{E}'_\bullet \arrow[r] & \mathcal{F} \arrow[r] & 0 \\
\mathcal{E}_\bullet \arrow[r] \arrow[u, two heads, "\varphi'"] \arrow[d, two heads, "\varphi''"'] & \mathcal{F} \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
\mathcal{E}''_\bullet \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Proof

Use Lemma 2 to build $\mathcal{E}_\bullet$.

$\square$

<u>Duality:</u> $K^\circ(X) \longrightarrow K_\circ(X)$

$E \mapsto$ sheaf of sections.

<u>Theorem</u> X non-singular variety

$\Rightarrow K^\circ(X) \xrightarrow{\cong} K_\circ(X)$ iso.

<u>Notation:</u> $K(X) = K^\circ(X) = K_\circ(X)$

<u>Proof of Theorem</u>

Given coherent $\mathcal{O}_X$-module $\mathcal{F}$,

choose resolution $\mathcal{E}_\bullet \to \mathcal{F} \to 0$ and define

$$\mu(\mathcal{F}) = \sum_{i \geq 0} (-1)^i [\mathcal{E}_i] \in K^\circ(X).$$

Then $\mu(\mathcal{F}) \mapsto [\mathcal{F}]$ under $K^\circ(X) \to K_\circ(X)$,

so this map is at least surjective.

$\mu(\mathcal{F})$ is well defined:

Given $\mathcal{E}'_\bullet \to \mathcal{F} \to 0$ and $\mathcal{E}''_\bullet \to \mathcal{F} \to 0$,

$\exists\ \mathcal{E}_\bullet \to \mathcal{F} \to 0$ with surjective

$\varphi' : \mathcal{E}_\bullet \twoheadrightarrow \mathcal{E}'_\bullet$, $\varphi'' : \mathcal{E}_\bullet \to \mathcal{E}''_\bullet$:

$$\begin{tikzcd}
\mathcal{E}'_\bullet \arrow[d] & \mathcal{E}_\bullet \arrow[l, two heads, "\varphi'"'] \arrow[r, two heads, "\varphi''"] \arrow[d] & \mathcal{E}''_\bullet \arrow[d] \\
\mathcal{F} \arrow[r, equal] \arrow[d] & \mathcal{F} \arrow[r, equal] \arrow[d] & \mathcal{F} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

<u>Show:</u> $\sum_{i\geq 0} (-1)^i [\mathcal{E}'_i] = \sum_{i \geq 0} (-1)^i [\mathcal{E}_i] \in K^0(X)$.

$K_\bullet = \operatorname{Ker}(\mathcal{E}_\bullet \to \mathcal{E}'_\bullet)$ complex of locally free $\mathcal{O}_X$-modules.

$$\begin{tikzcd}
0 \arrow[r] & K_\bullet \arrow[r] \arrow[d] & \mathcal{E}_\bullet \arrow[r] \arrow[d] & \mathcal{E}'_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r, equal] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

LES on homology groups $\Rightarrow K_\bullet$ exact.

$$\sum_{i\geq 0} (-1)^i [\mathcal{E}_i] - \sum_{i\geq 0} (-1)^i [\mathcal{E}'_i] = \sum_{i\geq 0} (-1)^i [K_i] = 0.$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules.

$\exists$ short exact seq. of resolutions:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}'_\bullet \arrow[r] \arrow[d] & \mathcal{E}_\bullet \arrow[r] \arrow[d] & \mathcal{E}''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

$\Rightarrow \mu(\mathcal{F}) = \mu(\mathcal{F}') + \mu(\mathcal{F}'')$.

This shows that μ defines a group homomorphism

$$K_0(X) \longrightarrow K^0(X), \quad [\mathcal{F}] \mapsto \mu(\mathcal{F})$$

which is inverse to the duality map $K^0(X) \to K_0(X)$.

$\square$

Facts about coherent sheaves

X scheme, $U \subseteq X$ open subscheme.

(1) $\mathcal{F}$ coherent $\mathcal{O}_U$-module $\Rightarrow$
$\exists$ coh. $\mathcal{O}_X$-module $\tilde{\mathcal{F}}$ s.t. $\tilde{\mathcal{F}}|_U \cong \mathcal{F}$.

(2) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$
short exact of coh. $\mathcal{O}_U$-modules
$\Rightarrow \exists \; 0 \to \tilde{\mathcal{F}}' \to \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'' \to 0$
restricting to $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$.

(3) $\mathcal{F}_1, \mathcal{F}_2$ coh. $\mathcal{O}_X$-modules such that
$\mathcal{F}_1|_U \cong \mathcal{F}_2|_U$ as $\mathcal{O}_U$-modules.
$\Rightarrow \exists$ coherent $\mathcal{O}_X$-module $\mathcal{G}$ with
$\mathcal{O}_X$-hom. $\mathcal{G} \to \mathcal{F}_1$, $\mathcal{G} \to \mathcal{F}_2$
restricting to $\mathcal{G}|_U \xrightarrow{\cong} \mathcal{F}_i|_U$.

__Exact sequence__

X scheme, $i: Z \subseteq X$ closed subscheme,

$j: U = X \setminus Z \subseteq X$ open. Then

$$K_0(Z) \xrightarrow{i_*} K_0(X) \xrightarrow{j^*} K_0(U) \to 0 \text{ is exact.}$$

__Proof__ (from Fulton-Lang)

Composition $= 0 \Rightarrow$

$$K_0(X)/i_* K_0(Z) \xrightarrow{\varphi} K_0(U) \quad \text{well def.}$$

Given $\mathcal{F}$ coherent $\mathcal{O}_U$-module,
choose $\tilde{\mathcal{F}}$ coherent $\mathcal{O}_X$-module
such that $\tilde{\mathcal{F}}|_U \cong \mathcal{F}$.

Set $\varphi(\mathcal{F}) = [\tilde{\mathcal{F}}] \in K_0(X)/i_* K_0(Z)$.

__well defined__:

Assume $\tilde{\mathcal{F}}|_U = \mathcal{F}$ and $\mathcal{G}|_U = \mathcal{F}$.

WLOG $\exists\ \mathcal{G} \to \tilde{\mathcal{F}}$ s.t. $\mathcal{G}|_U \xrightarrow{\cong} \tilde{\mathcal{F}}|_U$.

$$0 \to \mathcal{K} \to \mathcal{G} \to \tilde{\mathcal{F}} \to \mathcal{C} \to 0$$

$$[\mathcal{G}] - [\tilde{\mathcal{F}}] = [\mathcal{K}] - [\mathcal{C}] \in i_* K_0(Z).$$

<u>Note:</u> $\mathcal{K}|_U = 0 \Rightarrow I_Z^N \cdot \mathcal{K} = 0$

$$\Rightarrow [\mathcal{K}] = \sum_{j=0}^{N} \left[I_Z^j \mathcal{K} / I_Z^{j+1} \mathcal{K} \right] \in i_* K_0(Z).$$

<u>Inverse to φ:</u>

$$K_0(U) \longrightarrow K_0(X)/i_* K_0(Z)$$

$$[\mathcal{F}] \longmapsto \varphi(\mathcal{F})$$

<u>well defined:</u>

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

exact of coh. $\mathcal{O}_U$-module.

Extend to exact seq. of coh. $\mathcal{O}_X$-mods:

$$0 \to \tilde{\mathcal{F}}' \to \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'' \to 0$$

$$\begin{aligned} \varphi(\mathcal{F}) &= [\tilde{\mathcal{F}}] \\ &= [\tilde{\mathcal{F}}'] + [\tilde{\mathcal{F}}''] \\ &= \varphi(\mathcal{F}') + \varphi(\mathcal{F}''). \end{aligned}$$

$\square$