Cor $f: X \to Y$ morphism,
Y non-singular variety,
X CM of pure dim.
$V \subseteq Y$ closed subscheme.

Assume V is CM of pure dim.
and $f^{-1}(V)$ has pure dim. $=$
$$\dim(V) + \dim(X) - \dim(Y).$$

Then $f^{-1}(V)$ is CM and
$$f^*[\mathcal{O}_V] = [\mathcal{O}_{f^{-1}(V)}] \in K_0(X).$$

Proof

$$X \overset{\gamma}{\hookrightarrow} X \times Y \xrightarrow{\pi} Y$$

$\gamma(x) = (x, f(x))$ regular embedding,
since γ section to pr_X.

π flat and $\pi^{-1}(V) = X \times V$ is CM of pure dim.

$$\begin{aligned} f^*[\mathcal{O}_V] &= \gamma^* \pi^* [\mathcal{O}_V] \\ &= \gamma^* [\mathcal{O}_{X \times V}] \\ &= [\mathcal{O}_{f^{-1}(V)}]. \end{aligned}$$

□

Chern character

X alg. scheme / K.

$$A^*(X)_{\mathbb{Q}} = A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

E vector bundle of rank r.

Chern roots: $\alpha_1, \dots, \alpha_r$.

Chern classes: $c_i = c_i(E) = e_i(\alpha_1, \dots, \alpha_r)$.

$$ch(E) = \sum_{i=1}^{r} \exp(\alpha_i) \in A^*(X)_{\mathbb{Q}}$$

$$= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

Note:

$ch: K^0(X) \longrightarrow A^*(X)_{\mathbb{Q}}$ ring hom.

Todd class

$$Q(x) = \frac{x}{1-\exp(-x)}$$

$$= 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

where B_k = Bernoulli number (with right conventions!)

$$td(E) = \prod_{i=1}^{r} Q(\alpha_i) \in A^*(X)_{\mathbb{Q}}$$

$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1 c_2) + \cdots$$

Note:

$td : K^\circ(X) \longrightarrow A^*(X)_{\mathbb{Q}}^\times$ group hom.

<u>Example</u> $\operatorname{rank}(E) = r$.

$$\frac{c_r(E)}{td(E)} = \sum_{p=0}^{r} (-1)^p \, ch(\Lambda^p E^\vee)$$

$\Lambda^p E$ has Chern roots

$$\{\alpha_{i_1} + \cdots + \alpha_{i_p} \mid 1 \le i_1 < \cdots < i_p \le r\}$$

$$\text{RHS} = \sum_{p=0}^{r} (-1)^p \sum_{i_1 < \cdots < i_p} \exp(-\alpha_{i_1} - \cdots - \alpha_{i_p})$$

$$= \prod_{i=1}^{r} (1 - \exp(-\alpha_i))$$

$$= c_r(E) \prod_{i=1}^{r} \frac{1 - \exp(-\alpha_i)}{\alpha_i}$$

Example

$\mathbb{P}^n$:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

$$td(\mathcal{O}_{\mathbb{P}^n}) = Q(0) = 1.$$

$$td(T_{\mathbb{P}^n}) = td(\mathcal{O}(1))^{n+1}$$

$$= \left(\frac{x}{1-e^{-x}}\right)^{n+1} \in A(\mathbb{P}^n)_{\mathbb{Q}}.$$

where $x = c_1(\mathcal{O}(1))$.

Grothendieck-Riemann-Roch Theorem

$f: X \longrightarrow Y$ proper morphism of non-singular varieties.

For $\alpha \in K(X)$:

$$ch(f_*(\alpha)) \cdot td(T_Y) = f_*(ch(\alpha) \cdot td(T_X))$$

$$\in A^*(Y)_{\mathbb{Q}}$$

$$\begin{tikzcd}
K(X) \arrow[r, "f_*"] \arrow[d, "\tau_X"'] & K(Y) \arrow[d, "\tau_Y"] \\
A(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

$$\tau_X(\alpha) = ch(\alpha) \cdot td(T_X)$$

Hirzebruch's formula.

X complete non-singular variety.
E vector bundle. Then

$$\chi(X,E) = \int_X ch(E)\cdot td(T_X)$$

$f : X \longrightarrow \{pt\}$

$$\begin{tikzcd} K(X) \arrow[r, "\chi(X,-)"] \arrow[d, "\tau_X"'] & K(pt) \arrow[d, "\cap"] \\ A(X)_{\mathbb{Q}} \arrow[r, "\int_X"] & A(pt)_{\mathbb{Q}} \end{tikzcd}$$

Special case

$$\chi(X,\mathcal{O}_X) = \int_X td(T_X).$$

Example

$f : \mathbb{P}^n \longrightarrow \mathrm{Spec}(K).$

$\alpha = [\mathcal{O}(m)] \in K(\mathbb{P}^n).$

$$\chi(\mathbb{P}^n, \mathcal{O}(m)) = \int_{\mathbb{P}^n} ch(\mathcal{O}(m)) \cdot td(T_{\mathbb{P}^n})$$

$$LHS = \binom{n+m}{m}$$

$$x = c_1(\mathcal{O}(1)) \in A^1(\mathbb{P}^n)$$

$$td(T_{\mathbb{P}^n}) = \left(\frac{x}{1-e^{-x}}\right)^{n+1}$$

$$\begin{aligned} RHS &= \int_{\mathbb{P}^n} e^{mx} \left(\frac{x}{1-e^{-x}}\right)^{n+1} \\ &= \mathrm{coeff}\left(\frac{e^{mx}\, x^{n+1}}{(1-e^{-x})^{n+1}}, \; x^n\right) \\ &= \mathrm{Res}\left(\frac{e^{mx}}{(1-e^{-x})^{n+1}}, 0\right) \\ &= \frac{1}{2\pi i} \oint \frac{e^{mx}}{(1-e^{-x})^{n+1}}\, dx \end{aligned}$$

$$y = 1 - e^{-x}.$$

$$e^{x} = (1-y)^{-1}$$

$$dy = e^{-x}\,dx = (1-y)\,dx$$

$$\frac{1}{2\pi i} \oint \frac{e^{mx}}{(1-e^{-x})^{n+1}}\,dx =$$

$$\frac{1}{2\pi i} \oint \frac{(1-y)^{-m-1}}{y^{n+1}}\,dy =$$

$$\mathrm{Res}\left(\frac{1}{(1-y)^{m+1}\, y^{n+1}}, 0 \right) =$$

$$\mathrm{coeff}\left(\frac{1}{(1-y)^{m+1}}, y^{n} \right) = \binom{n+m}{n}$$

Example

$f = \pi_1 : Y \times \mathbb{P}^n \longrightarrow Y$ projection.

Product map:

$$\times : K(Y) \otimes K(\mathbb{P}^n) \longrightarrow K(Y \times \mathbb{P}^n)$$
$$\alpha \otimes \beta \longmapsto \pi_1^*(\alpha) \cdot \pi_2^*(\beta)$$

Fact: surjective.

$$K(Y \times \mathbb{P}^{n-1}) \longrightarrow K(Y \times \mathbb{P}^n) \longrightarrow K(Y \times \mathbb{A}^n) = K(Y)$$

$$K(Y) \otimes K(\mathbb{P}^n) \xrightarrow{\times} K(Y \times \mathbb{P}^n) \xrightarrow{\pi_{1*}} K(Y)$$
$$\alpha \otimes \beta \longmapsto \pi_{1*}(\pi_1^*(\alpha) \cdot \pi_2^*(\beta))$$
$$= \alpha \cdot \pi_{1*} \pi_2^*(\beta)$$
$$= \chi(\beta) \alpha$$

$$\begin{tikzcd} Y \times \mathbb{P}^n \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ Y \arrow[r] & pt \end{tikzcd}$$

$$A(Y) \otimes A(\mathbb{P}^n) \xrightarrow{\times} A(Y \times \mathbb{P}^n) \xrightarrow{\pi_{1*}} K(Y)$$
$$c \otimes d \longmapsto c \cdot \pi_{1*} \pi_2^*(d)$$
$$= \left(\int d\right) \cdot c$$

$$\begin{tikzcd}
K(Y) \otimes K(\mathbb{P}^n) \arrow[r, "\times"] \arrow[d, "\tau_Y \otimes \tau_{\mathbb{P}^n}"] & K(Y \times \mathbb{P}^n) \arrow[r, "f_*"] \arrow[d, "\tau_{Y \times \mathbb{P}^n}"] & K(Y) \arrow[d, "\tau_Y"] \\
A(Y)_{\mathbb{Q}} \otimes A(\mathbb{P}^n)_{\mathbb{Q}} \arrow[r, "\times"] & A(Y \times \mathbb{P}^n)_{\mathbb{Q}} \arrow[r, "f_*"] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

LHS commutes: $td(T_Y) \times td(T_{\mathbb{P}^n}) = td(T_{Y \times \mathbb{P}^n})$

Enough: outer diagram commutes.

Reduces to $f: \mathbb{P}^n \longrightarrow \{pt\}$.

<u>General case</u>:

factor $f: X \longrightarrow Y$:

$$X \underset{\subseteq}{\xrightarrow{p}} Y \times \mathbb{P}^n \xrightarrow{q} Y$$

$$\begin{tikzcd}
K(X) \arrow[r, "p_*"] \arrow[d, "\tau_X"] & K(Y \times \mathbb{P}^n) \arrow[r, "q_*"] \arrow[d, "\tau_{Y \times \mathbb{P}^n}"] & K(Y) \arrow[d, "\tau_Y"] \\
A(X)_{\mathbb{Q}} \arrow[r, "p_*"'] & A(Y \times \mathbb{P}^n)_{\mathbb{Q}} \arrow[r, "q_*"'] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

Hard part: $f: X \hookrightarrow Y$ closed embedding.

Riemann-Roch for curves

X complete non-singular curve.

$g = 1 - \chi(\mathcal{O}_X) = \dim_k H^1(X, \mathcal{O}_X)$ genus.

$K = c_1(\omega_X) = -c_1(T_X)$ canonical div.

$td(T_X) = 1 - \frac{1}{2}K \in A(X)_{\mathbb{Q}}$.

HRR $\Rightarrow$ $1-g = \chi(\mathcal{O}_X) = \int td(T_X) = -\frac{1}{2}\deg(K)$.

D any divisor on X.

$\ell(D) = \dim_k H^0(X, \mathcal{O}_X(D))$.

$\ell(D) > 0 \iff D \equiv$ effective divisor.

Serre duality:

$$H^i(X, \mathcal{O}_X(D)) = H^{1-i}(X, \mathcal{O}_X(K-D))$$

$\Rightarrow \chi(X, \mathcal{O}_X(D)) = \ell(D) - \ell(K-D)$.

HRR $\Rightarrow$

$$\ell(D) - \ell(K-D) = \int_X ch(\mathcal{O}_X(D)) \cdot td(T_X)$$

$$= \int_X \exp(D) \cdot (1 - \tfrac{1}{2}K)$$

$$= \deg(D) - \tfrac{1}{2}\deg(K)$$

$$= \deg(D) + 1 - g$$

<u>Note:</u>

$\deg(D) < 0 \Rightarrow D \not\equiv$ effective divisor

$\Rightarrow \ell(D) = 0.$

$\deg(D) > \deg(K) \Rightarrow \ell(K-D) = 0$

$\Rightarrow \ell(D) = \deg(D) + 1 - g$