Surfaces

X complete non-singular surface.

$c_i = c_i(T_X)$.

$$\chi(X, \mathcal{O}_X) = \int_X td(T_X) = \frac{1}{12}\int_X (c_1^2 + c_2).$$

E vector bundle of rank r.

Chern classes: $d_i = c_i(E)$, $1 \leq i \leq r$.

$$\chi(X, E) = \int_X ch(E) \cdot td(T_X)$$

$$= \int_X \left(r + d_1 + \tfrac{1}{2}(d_1^2 - 2d_2)\right) \cdot \left(1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2)\right)$$

$$= \frac{1}{2}\int_X \left(d_1^2 - 2d_2 + c_1 d_1\right) + r\,\chi(X, \mathcal{O}_X).$$

$E = \mathcal{O}_X(D)$ line bundle:

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{2}\int_X (D \cdot D - D \cdot K) + \chi(X, \mathcal{O}_X),$$

$K = c_1(\omega_X) = -c_1$ canonical divisor.

$D \subseteq X$ effective Cartier divisor.

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

$$\begin{aligned} p_a(D) &= 1 - \chi(X, \mathcal{O}_D) \\ &= 1 - \chi(X, \mathcal{O}_X) + \chi(X, \mathcal{O}_X(-D)) \\ &= 1 + \frac{1}{2} \int_X (D \cdot D + D \cdot K) \end{aligned}$$

<u>$X = \mathbb{P}^2$:</u>

$\omega_X = \mathcal{O}(-3)$

$K = -3h$, $h = [H]$ hyperplane class.

$D = dh$:

$$\begin{aligned} \chi(\mathbb{P}^2, \mathcal{O}(d)) &= \frac{1}{2} \int (D \cdot D - D \cdot K) + \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \\ &= \frac{1}{2}(d^2 + 3d) + 1 = \frac{1}{2}(d+1)(d+2). \end{aligned}$$

$C \subseteq \mathbb{P}^2$ curve of degree d.

$$p_a(C) = 1 + \frac{1}{2}(d^2 - 3d) = \frac{1}{2}(d-1)(d-2).$$

$\underline{X = \mathbb{P}^1 \times \mathbb{P}^1}$:

$$A(X) = \mathbb{Z}[s,t] / \langle s^2, t^2 \rangle$$

$$s = [0 \times \mathbb{P}^1], \quad t = [\mathbb{P}^1 \times 0].$$

$$T_X = \pi_1^* T_{\mathbb{P}^1} \oplus \pi_2^* T_{\mathbb{P}^1} = \mathcal{O}_X(2s) \oplus \mathcal{O}_X(2t).$$

$$K = -2(s+t).$$

$$\begin{aligned} td(T_X) &= td(\mathcal{O}_X(2s)) \cdot td(\mathcal{O}_X(2t)) \\ &= (1+s)(1+t) = 1 + s + t + st. \end{aligned}$$

$$\chi(X, \mathcal{O}_X) = \int_X td(T_X) = 1.$$

$D = ms + nt$:

$$\chi(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(ms+nt))$$

$$= \frac{1}{2}\int_X (D\cdot D - D\cdot K) + \chi(X, \mathcal{O}_X)$$

$$= 1 + \frac{1}{2}\int_X \left((ms+nt)^2 + 2(ms+nt)(s+t)\right)$$

$$= 1 + \frac{1}{2}(2mn + 2m + 2n)$$

$$= (m+1)(n+1).$$

$C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ curve of bidegree (m,n).

$$p_a(C) = 1 + \frac{1}{2}\int_X (D\cdot D + D\cdot K)$$

$$= 1 + \frac{1}{2}\int_X \left((ms+nt)^2 - 2(ms+nt)(s+t)\right)$$

$$= 1 + \frac{1}{2}(2mn - 2m - 2n)$$

$$= (m-1)(n-1).$$

Topological filtration of $K_0(X)$

X alg. scheme.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

$\operatorname{Supp}(\mathcal{F}) = \{ p \in X \mid \mathcal{F}_p \neq 0 \} \subseteq X$ closed.

$$F_k K_0(X) = \langle [\mathcal{F}] \mid \dim \operatorname{Supp}(\mathcal{F}) \leq k \rangle \subseteq K_0(X).$$

$K^0(X)$ - submodule.

$$Gr_k K_0(X) = F_k K_0(X) / F_{k-1} K_0(X).$$

<u>Goal</u>: $Gr_k K_0(X) \otimes \mathbb{Q} \cong A_k(X) \otimes \mathbb{Q}$.

$f : X \longrightarrow Y$ proper morphism :

$$\operatorname{Supp}(R^i f_* \mathcal{F}) \subseteq f(\operatorname{Supp}(\mathcal{F})).$$

$$f_*(F_k K_0(X)) \subseteq F_k K_0(Y).$$

$$\therefore f_* : Gr_k K_0(X) \longrightarrow Gr_k K_0(Y).$$

Lemma:

$$F_k K_0(X) = \left\langle [\mathcal{O}_V] \;\middle|\; \begin{array}{l} V \subseteq X \text{ closed irred} \\ \text{subvar, } \dim V \leq k \end{array} \right\rangle$$

Proof

$\mathcal{F}$ coherent $\mathcal{O}_X$-module,

$\dim \operatorname{Supp}(\mathcal{F}) \leq k$.

Show: $[\mathcal{F}] \in$ RHS.

WLOG: $X = \operatorname{Supp}(\mathcal{F})$.

Let $V \subseteq X$ subvar. of dim. k.

$$K_0(V) \longrightarrow K_0(X) \longrightarrow K_0(X-V) \longrightarrow 0$$

$$K_0(\overline{X-V}) \longrightarrow K_0(X-V) \longrightarrow 0$$

$$\Rightarrow K_0(X) = K_0(V) + K_0(X-V).$$

WLOG: $X = V$ irred. variety.

$U \subseteq X$ open affine, $U = \mathrm{Spec}(R)$.

$\mathcal{F}|_U = \tilde{M}$, M f.g. R-module.

$$\mathcal{F}_0 = M \otimes_R K(R) \cong K(R)^{\oplus N}.$$

$$\exists\, 0 \neq f \in R: \; M_f \cong R_f^{\oplus N}.$$

Replace U with $\mathrm{Spec}(R_f)$:

WLOG: $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus N}$.

$$K_0(X-U) \longrightarrow K_0(X) \longrightarrow K_0(U) \longrightarrow 0$$

$$[\mathcal{F}] - N[\mathcal{O}_V] \longmapsto 0$$

$\therefore \; [\mathcal{F}] - N[\mathcal{O}_V] \in F_{k-1} K_0(X)$.

generated by subvarieties of dim $\leq k-1$

by induction on k.

$\square$

Universal property of $A_*(-)$

$\mathscr{C}$ category of alg. schemes $/K$
and proper morphisms.

$H : \mathscr{C} \longrightarrow \underline{Ab}$ functor.

Assume that each irred. variety
V has a class $cl(V) \in H(V)$,
such that:

(1) $f : V \longrightarrow W$ surjective + proper $\Rightarrow$

$$f_*(cl(V)) = \deg(V/W)\, cl(W) \in H(W)$$

Then $cl : Z_* \longrightarrow H$ nat. trans.

(2) X normal variety, $f : X \longrightarrow \mathbb{P}^1$ dominant

$$\Rightarrow \quad cl([f^{-1}(0)]) = cl([f^{-1}(\infty)]).$$

<u>Fact:</u> Then $cl : A_* \longrightarrow H$ nat. trans.

$Gr_k\, K_0(-) : \mathcal{C} \longrightarrow \underline{Ab}$ functor.

<u>Class of irred variety:</u>

$$\varphi(V) = [\mathcal{O}_V] \in Gr_k\, K_0(X),$$
$$k = \dim(V).$$

<u>Prop</u> $f: X \longrightarrow Y$ proper morphism of irred. varieties $\Rightarrow$

<u>Claim</u>: $f: X \longrightarrow Y$ surj. proper morphism of irred. varieties $\Rightarrow$

$$f_*[\mathcal{O}_X] = \deg(X/Y)\,[\mathcal{O}_Y] \in Gr_k\, K_0(Y),$$
$$k = \dim(X).$$

Proof

WLOG: $\dim(Y) = k$.

$U = \mathrm{Spec}(R) \subseteq Y$ open affine.

$W = \mathrm{Spec}(S) \subseteq f^{-1}(U) \subseteq X$ open.

$S \otimes_R K(R)$ Artinian domain $\Rightarrow$ field

$$\Rightarrow S \otimes_R K(R) = K(S) \cong K(R)^{\oplus d}$$

$$d = \deg(X/Y).$$

$\exists\, 0 \neq f \in R:\quad S_f \cong R_f^{\oplus d}$.

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \\ W_f \arrow[u, "\cup"'] \arrow[r, "g", "\text{finite}"'] & U_f \arrow[u, "\cup"'] \end{tikzcd}$$

$$\left(R^i f_* \mathcal{O}_X\right)\big|_{U_f} = R^i g_* \mathcal{O}_{W_f} = \begin{cases} \mathcal{O}_{U_f}^{\oplus d} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

$\Rightarrow f_*[\mathcal{O}_X] = d\,[\mathcal{O}_Y] \in \mathrm{Gr}_* K_0(Y)$.

$\square$