R Noeth ring, M f.g. R-module.

$\mathrm{Supp}(M) = \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0 \}$.

$\mathrm{Ann}(M) = \{ f \in R \mid f \cdot M = 0 \}$.

$\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$:

$$M_\mathfrak{p} = 0 \iff \exists f \in R \setminus \mathfrak{p} : f \cdot M = 0$$
$$\iff \mathrm{Ann}(M) \not\subseteq \mathfrak{p}.$$

$$\sqrt{\mathrm{Ann}(M)} = \bigcap_{\mathfrak{p} \supseteq \mathrm{Ann}(M)} \mathfrak{p}$$

Let $\mathfrak{p}$ be a min. prime over $\mathrm{Ann}(M)$.
i.e. $V(\mathfrak{p})$ irred. comp. of $\mathrm{Supp}(M)$.

Then $\mathfrak{p}_\mathfrak{p} \subseteq R_\mathfrak{p}$ unique min. prime over $\mathrm{Ann}(M_\mathfrak{p})$

$\Rightarrow \mathfrak{p}_\mathfrak{p}^N \cdot M_\mathfrak{p} = 0$

$\Rightarrow M_\mathfrak{p}$ f.g. module over $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^N$

$\Rightarrow \mathrm{length}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$.

$\mathcal{F}$ coherent $\mathcal{O}_X$-module.

For $V \subseteq X$ irred. closed subvariety:

$\mathcal{F}_V$ = stalk at generic point of V.

$\mathcal{O}_{V,X}$ - module.

Assume $\dim \mathrm{Supp}(\mathcal{F}) \leq \dim(V)$.

$m_V(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_{V,X}}(\mathcal{F}_V) < \infty$.

Def For $\dim \mathrm{Supp}(\mathcal{F}) \leq k$, set

$$Z_k(\mathcal{F}) = \sum_{\dim(V)=k} m_V(\mathcal{F})[V] \in Z_k(X).$$

Example X scheme, $\dim(X) = k$

$$\Rightarrow [X] = Z_k(\mathcal{O}_X) \in Z_k(X).$$

Def $\varphi : Z_k(X) \longrightarrow Gr_k K_0(X)$.

$$[V] \longmapsto [\mathcal{O}_V]$$

This is a nat. trans. $\varphi : Z_* \longrightarrow Gr_* K_0(-)$ on cat. of alg. schemes with proper morphisms.

Lemma Assume $\dim \mathrm{Supp}(\mathcal{F}) \le k$. Then

$$\varphi(Z_k(\mathcal{F})) = [\mathcal{F}] \in Gr_k K_0(X).$$

Proof

WLOG: $X = \mathrm{Supp}(\mathcal{F})$.

$$[\mathcal{F}] - \sum_{\dim(V)=k} c_V [\mathcal{O}_V] \in F_{k-1} K_0(X)$$

for <u>some</u> constants $c_V \in \mathbb{Z}$.

Given $V \subseteq X$ of dim. k, have group hom.

$$K_0(X) \longrightarrow \mathbb{Z}, \quad \mathcal{F} \longmapsto m_V(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_{V,X}}(\mathcal{F}_V).$$

$$m_V(\mathcal{O}_V) = 1 \Rightarrow c_V = m_V(\mathcal{F}).$$

$\square$

<u>Cor</u> φ defined on $A_k(X)$:

$$\varphi : A_k(X) \longrightarrow Gr_k K_0(X)$$
$$[V] \longmapsto [\mathcal{O}_V].$$

<u>Proof</u>

Use univ. property of A_*:

Let $f: X \longrightarrow \mathbb{P}^1$ be dominant,

X (normal) variety.

<u>Show:</u> $\varphi([f^{-1}(0)]) = \varphi([f^{-1}(\infty)])$

f flat. Let $t \in \mathbb{P}^1$.

$$\begin{aligned}\varphi([f^{-1}(t)]) &= \varphi(Z_k(\mathcal{O}_{f^{-1}(t)})) \\ &= [\mathcal{O}_{f^{-1}(t)}] \\ &= f^*[\mathcal{O}_{\{t\}}] \\ &= f^*([\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\mathbb{P}^1}(-1)])\end{aligned}$$

independent of t.

$\square$

FGRR Theorem (extract)

For each alg. scheme X/K,
$\exists$ group hom. $\tau_X : K_0(X) \longrightarrow A_*(X)_{\mathbb{Q}}$
such that:

(1) $f : X \longrightarrow Y$ proper, $\alpha \in K_0(X)$

$$\Rightarrow \quad f_*(\tau_X(\alpha)) = \tau_Y(f_*(\alpha)) \in A_*(Y)_{\mathbb{Q}}.$$

(2) $\beta \in K^\circ(X)$, $\alpha \in K_0(X)$

$$\Rightarrow \quad \tau_X(\beta \cdot \alpha) = ch(\beta) \cap \tau_X(\alpha) \in A_*(X)_{\mathbb{Q}}.$$

(3) $f : X \hookrightarrow Y$ regular embedding,
$\alpha \in K_0(Y)$

$$\Rightarrow \quad \tau_X(f^!(\alpha)) = td(N_X Y)^{-1} \cdot f^*(\tau_Y(\alpha)).$$

(4) $V \subseteq X$ closed irred. subvar.

$$\Rightarrow \quad \tau_X([\mathcal{O}_V]) = [V] + \text{lower terms}.$$

Example

X non-sing. variety.

$$\tau_X(\alpha) = ch(\alpha)\cdot td(T_X), \quad \alpha \in K(X).$$

(1) is GRR (Y also non-sing).

(2) $$\tau_X(\beta\cdot\alpha) = ch(\beta\cdot\alpha)\cdot td(T_X)$$
$$= ch(\beta)\cdot ch(\alpha)\cdot td(T_X)$$
$$= ch(\beta)\cdot \tau_X(\alpha).$$

(3) $f: X \hookrightarrow Y$ closed embedding of non-sing. varieties.
Then f is a regular embedding.

$$0 \to T_X \to f^*T_Y \to N_XY \to 0$$

$\alpha \in K(Y)$.

$$f^*(\tau_Y(\alpha)) = f^*(ch(\alpha))\cdot f^*(td(T_Y))$$
$$= ch(f^*(\alpha))\cdot td(f^*T_Y)$$
$$= ch(f^*(\alpha))\cdot td(T_X)\cdot td(N_XY)$$
$$= td(N_XY)\cdot \tau_X(f^*(\alpha)).$$

(4) X non-sing. variety,

$i: V \subseteq X$ closed irred. subvar.

$$\begin{aligned} \tau_X([\mathcal{O}_V]) &= \tau_X(i_*([\mathcal{O}_V])) \\ &= i_*(\tau_V([\mathcal{O}_V])) \end{aligned}$$

<u>V non-singular</u>:

$$\begin{aligned} \tau_X([\mathcal{O}_V]) &= i_*(ch([\mathcal{O}_V]) \cdot td(T_V)) \\ &= i_*(td(T_V) \cap [V]) \\ &= [V] + \text{lower terms.} \end{aligned}$$

Note also:

$$\begin{aligned} ch([\mathcal{O}_V]) &= \tau_X([\mathcal{O}_V]) \cdot td(T_X)^{-1} \\ &= i_*(td(T_V) \cdot td(i^*(T_X))^{-1} \cap [V]) \\ &= i_*(td(N_V X)^{-1} \cap [V]) \end{aligned}$$

$$0 \to T_V \to i^* T_X \longrightarrow N_V X \longrightarrow 0$$

X non-sing, $V \subseteq X$ any closed subvariety.

$U = X - V_{sing} \subseteq X$ open.

$V \cap U$ non-singular.

$$\begin{tikzcd}
K(X) \arrow[r, "\tau_X"] \arrow[d] & A_*(X)_{\mathbb{Q}} & \supseteq & A_k(X)_{\mathbb{Q}} \arrow[d, "\cong"] & {[V]} \\
K(U) \arrow[r, "\tau_U"] & A_*(U)_{\mathbb{Q}} \arrow[from=u] & \supseteq & A_k(U)_{\mathbb{Q}} & {[V \cap U]}
\end{tikzcd}$$

$$\tau_U([\mathcal{O}_{V \cap U}]) = [V \cap U] + \text{lower terms}$$

$$\Rightarrow \tau_X([\mathcal{O}_V]) = [V] + \text{lower terms.}$$

Todd class of alg. scheme

$$Td(X) = \tau_X([\mathcal{O}_X]) \in A_*(X)_{\mathbb{Q}}.$$

X non-sing. variety $\Rightarrow$ $Td(X) = td(T_X)$.

<u>Cor</u> X complete alg. scheme, E vector bundle

$$\Rightarrow \chi(X, E) = \int_X ch(E) \cap Td(X)$$

$$\chi(X, \mathcal{O}_X) = \int_X Td(X)$$

<u>Proof</u>

$$\begin{tikzcd} K_0(X) \arrow[r, "\chi"] \arrow[d, "\tau_X"'] & K_0(pt) \arrow[d] \\ A_*(X)_{\mathbb{Q}} \arrow[r, "\int_X"'] & A(pt)_{\mathbb{Q}} \end{tikzcd}$$

$$\begin{aligned} \tau_X(E) &= \tau_X([E] \cdot [\mathcal{O}_X]) \\ &= ch([E]) \cap \tau_X([\mathcal{O}_X]) \\ &= ch(E) \cap Td(X). \end{aligned}$$

□

Cor X alg. scheme $\Rightarrow$

$$\tau_X : K_0(X)_{\mathbb{Q}} \xrightarrow{\cong} A_*(X)_{\mathbb{Q}}$$

isomorphism of groups.

Proof

$$K_0(X)_{\mathbb{Q}} \xrightarrow{\tau_X} A_*(X)_{\mathbb{Q}} \xrightarrow{\varphi} Gr_* K_0(X)_{\mathbb{Q}}$$

$$[\mathcal{O}_V] \longmapsto [V] + \text{lower terms}$$

$$[V] \longmapsto [\mathcal{O}_V]$$

Let $\alpha \in K_0(X)_{\mathbb{Q}}$, $\alpha \neq 0$.

Choose k s.t. $\alpha \in F_k K_0(X)_{\mathbb{Q}} - F_{k-1} K_0(X)_{\mathbb{Q}}$.

$$\alpha - \sum_{\dim(V)=k} c_V [\mathcal{O}_V] \in F_{k-1} K_0(X)_{\mathbb{Q}}.$$

$$\left.\begin{aligned} \varphi(\tau_X(\alpha)) &= \varphi(\tau_X(\textstyle\sum c_V[\mathcal{O}_V])) \\ &= \textstyle\sum c_V [\mathcal{O}_V] \\ &= \alpha \neq 0 \end{aligned}\right\} \text{mod } F_{k-1} K_0(X)_{\mathbb{Q}}$$

$$\in Gr_k K_0(X).$$

$\therefore$ $\tau_X \otimes \mathbb{Q}$ injective.

$\square$

Cor X non-singular variety.

$$ch : K(X)_{\mathbb{Q}} \xrightarrow{\cong} A(X)_{\mathbb{Q}}$$

isomorphism of $\mathbb{Q}$-algebras.

Redefine $A(X)_{\mathbb{Q}} = K(X)_{\mathbb{Q}}$?

Anything missing ?

Grading !

$$ch : K(X)_{\mathbb{Q}} \xrightarrow{\cong} A(X)_{\mathbb{Q}}$$

$$[\mathcal{O}_V] \longmapsto [V] + \text{lower terms.}$$

$$ch^{-1}([V]) = ???$$

Set $I_r = F_{n-r} K(X) \subseteq K(X)$, $n = \dim(X)$.

$$K(X) = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq I_{n+1} = 0.$$

Sometimes :

- $I_\bullet$ is a ring filtration: $I_p I_q \subseteq I_{p+q}$.
- $Gr\, K(X)_{\mathbb{Q}} \cong A(X)_{\mathbb{Q}}$ as graded rings.