Fact (Miller–Speyer)

G alg. group $/ k = \bar{k}$,

X transitive G-variety,

$\mathcal{F}, \mathcal{E}$ coherent $\mathcal{O}_X$-modules.

For $g \in G$, $g.\mathcal{F} = g_* \mathcal{F}$, $\quad g : X \longrightarrow X$.

$\exists$ dense open $U \subseteq X$:

$$\mathrm{Tor}_j^X(g.\mathcal{F}, \mathcal{E}) = 0 \quad \forall j > 0.$$

Consequence:

$$[g.\mathcal{F}] \cdot [\mathcal{E}] = \sum_{i \geq 0} (-1)^i [\mathrm{Tor}_j^X(g.\mathcal{F}, \mathcal{E})]$$

$$= [g.\mathcal{F} \otimes \mathcal{E}] \quad \in K(X).$$

<u>Cor</u> X non-singular variety $/ K = \bar{K}$

$\mathrm{char}(K) = 0$,

G <u>rational</u> algebraic group

$G \circlearrowright X$ transitive.

Then $I_r = F_{\dim(X) - r} K(X)_{\mathbb{Q}}$ is a ring filtration, and

$$\varphi : A(X)_{\mathbb{Q}} \xrightarrow{\cong} \mathrm{Gr}\, K(X)_{\mathbb{Q}} = \bigoplus_{r \geq 0} I_r / I_{r+1}$$

$$[V] \longmapsto [\mathcal{O}_V]$$

isomorphism of graded rings.

<u>Proof</u>

$V, W \subseteq X$ irred. closed subvars of codim. p, q. $[\mathcal{O}_V] \in I_p$, $[\mathcal{O}_W] \in I_q$.

<u>Show:</u>

- $[\mathcal{O}_V] \cdot [\mathcal{O}_W] \in I_{p+q}$ (ring filtration)
- $\varphi([V] \cdot [W]) = [\mathcal{O}_V] \cdot [\mathcal{O}_W]$ in I_{p+q} / I_{p+q+1}.

G rational $\Rightarrow$

$$[g.W] = [W] \in A(X) \qquad \forall g \in G.$$
$$[\mathcal{O}_{g.W}] = [\mathcal{O}_W] \in K(X)$$

Choose $g \in G$ such that

- $V \cap g.W$ disjoint union of irred. vars. of codim. $p+q$.
- $[V] \cdot [g.W] = [V \cap g.W]$
- $[\mathcal{O}_V] \cdot [\mathcal{O}_{g.W}] = [\mathcal{O}_{V \cap g.W}]$

Then $[\mathcal{O}_V] \cdot [\mathcal{O}_W] = [\mathcal{O}_{V \cap g.W}] \in I_{p+q}$.

$$\begin{aligned}
\varphi([V] \cdot [W]) &= \varphi([V \cap g.W]) \\
&= [\mathcal{O}_{V \cap g.W}] \\
&= [\mathcal{O}_V] \cdot [\mathcal{O}_W] \\
&= \varphi([V]) \cdot \varphi([W])
\end{aligned}$$

$\square$

Grassmannians $\quad K = \bar{K}$.

$X = Gr(m,n) = \{V \subseteq K^n \mid \dim(V) = m\}$.

Nonsingular, rational, projective.

$\dim(X) = m(n-m)$.

Plücker embedding: $X \hookrightarrow \mathbb{P}(\Lambda^m K^n)$

$V \longmapsto \Lambda^m V$.

$G = GL(n,K) \qquad G \circlearrowright K^n$ transitive

$G \circlearrowright X$ transitive.

Max. torus: $T \subseteq G$ diag. matrices.

T-fixed points:

$X^T = \{V \in X \mid t.V = V \ \forall t \in T\}$

Coordinate basis of K^n: $\{e_1, \dots, e_n\}$.

For $I \subseteq [n] = \{1,2,\dots,n\}$:

$E_I = \mathrm{Span}_K\{e_i \mid i \in I\} \subseteq K^n$.

$X^T = \{E_I \mid I \subseteq [n],\ \#I = m\}$.

Borel subgroup

$B \subseteq G$ upper ∇ matrices

Def: $\begin{bmatrix} n \\ m \end{bmatrix} = \{ I \subseteq [n] \mid \# I = m \}$

Given $I \in \begin{bmatrix} n \\ m \end{bmatrix}$:

Schubert cell: $\mathring{X}_I = B . E_I \subseteq X$

Schubert variety: $X_I = \overline{B . E_I} \subseteq X$.

Standard flag in K^n:

$F_\bullet = (0 = F_0 \subsetneq F_1 \subsetneq \cdots \subseteq F_n = K^n)$,

$F_p = \operatorname{Span}\{e_1, e_2, \ldots, e_p\}$.

Given $V \in X$, def. $I(V) \in \begin{bmatrix} n \\ m \end{bmatrix}$ by:

$I(V) = \{ i \in [n] \mid V \cap F_i \supsetneq V \cap F_{i-1} \}$.

Fact: $\mathring{X}_I = \{ V \in X \mid I(V) = I \} \cong \mathbb{A}^{|I|}$

where $|I| = \left(\sum_{j \in I} j \right) - \binom{m+1}{2}$

<u>Idea:</u> $V \in X$.

V = Row span of $A \in \mathrm{Mat}(m \times n, K)$.

WLOG: A in "leftward" RREF.

$$A = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 1 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & * & 1 & 0 \end{bmatrix}$$

$I(V) = \{ j \mid \text{pivot in col. } j \} = \{2, 5, 7\}$.

$\mathring{X}_I = B.E_I = \{ V \in X \mid \text{RREF has this form} \}$
$\cong \mathbb{A}^{|I|}$.

<u>Bruhat order:</u>

$I = \{ i_1 < \cdots < i_m \}$, $J = \{ j_1 < \cdots < j_m \} \in \binom{[n]}{m}$

$I \le J \Leftrightarrow i_p \le j_p$ for $1 \le p \le m$.

<u>Fact:</u>

$X_J = \overline{B.E_J} = \{ V \in X \mid I(V) \le J \} = \bigcup_{I \le J} \mathring{X}_I$.

$= \{ V \in X \mid \dim(V \cap F_{j_p}) \ge r,\ 1 \le p \le m \}$.

where $J = \{ j_1 < \cdots < j_p \}$.

<u>Note:</u> $I \le J \Leftrightarrow X_I \subseteq X_J$.

Cor $Y = X_{I_1} \cup X_{I_2} \cup \cdots \cup X_{I_\ell} \subseteq X$.

$\{[X_I] \mid X_I \subseteq Y\}$ generate $A_*(Y)$.

$\{\mathcal{O}_I = [\mathcal{O}_{X_I}] \mid X_I \subseteq Y\}$ gen. $K_\circ(Y)$

Proof

Choose I maximal with $X_I \subseteq Y$.

$Y' = Y \smallsetminus \mathring{X}_I$.

$$A_*(Y') \longrightarrow A_*(Y) \longrightarrow \underset{\mathbb{Z}}{\overset{}{A_*(\mathring{X}_I)}} \longrightarrow 0$$

$A_*(\mathring{X}_I) = \mathbb{Z}$

$\square$

Opposite Schubert varieties

$B^- \subseteq G$ lower $\triangle$ matrices. $B \cap B^- = T$.

$\mathring{X}^I = B^-.E_I$, $\quad X^I = \overline{B^-.E_I} \subseteq X$.

$w_\circ \in G$, $\quad w_\circ(e_i) = e_{n+1-i}$, $\quad 1 \leq i \leq n$.

$w_\circ.\mathring{X}^I = \mathring{X}_{I^\vee}$, $\quad I^\vee = \{n+1-i \mid i \in I\}$.

$\mathring{X}^I \cong \mathbb{A}^{\dim(X) - |I|}$

$X^J = \bigcup_{I \geq J} \mathring{X}^I$.

Fact (Ramanan, Ramanathan)
X_I is normal, CM, has rational sings.

Borel fixed-point theorem
G connected solvable LAG $/K=\overline{K}$,
$G \circlearrowright Y$, $Y \neq \emptyset$ complete variety.
$\Rightarrow Y^G \neq \emptyset$.

Lemma $X_I \cap X^J \neq \emptyset \Leftrightarrow \mathring{X}_I \cap \mathring{X}^J \neq \emptyset \Leftrightarrow J \leq I$.
In this case, $X_I \cap X^J$ proper intersection,
$\mathring{X}_I \cap \mathring{X}^J \subseteq X_I \cap X^J$ dense open.

Proof
$(X_I \cap X^J)^T = \{E_{I'} \mid J \leq I' \leq I\}$
(or use dim. conditions.)

Kleiman:
$\exists\, U \subseteq G$: $X_I \cap g.X^J$ proper $\forall g \in U$.

$BB^- \subseteq G$ dense open.

Choose $g = bb' \in U \cap BB^-$, $b \in B$, $b' \in B^-$.

$$\begin{aligned} X_I \cap g.X^J &= b.X_I \cap bb'.X^J \\ &= b.(X_I \cap X^J). \end{aligned}$$

$\square$

Cor $[X_I]\cdot[X^J] = [X_I \cap X^J] \in A(X)$

$$\mathcal{O}_I \cdot \mathcal{O}^J = [\mathcal{O}_{X_I \cap X^J}] \in K(X).$$

$$\mathcal{O}_I = [\mathcal{O}_{X_I}], \quad \mathcal{O}^J = [\mathcal{O}_{X^J}].$$

Cor $Y = X_{I_1} \cup \cdots \cup X_{I_\ell} \Rightarrow$

$\{[X_I] \mid X_I \subseteq Y\}$ basis of $A_*(Y)$

$\{\mathcal{O}_I \mid X_I \subseteq Y\}$ basis of $K_\circ(Y)$.

Proof

$K_\circ(Y) \longrightarrow K(X)$.

Show $\{\mathcal{O}_I \mid I \in \binom{[n]}{m}\}$ lin. indep. in $K(X)$.

Note: $X_I \cap X^I = \{E_I\}$

(T-stable, $\dim = 0 \Rightarrow X_I \cap X^I \subseteq X^T$.)

$$\chi(\mathcal{O}_I \cdot \mathcal{O}^J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

$\square$

Richardson variety:

$X_I \cap X^J$ (when $\neq \emptyset$.)

Fact: $X_I \cap X^J$ irred. rational variety, normal, CM, rat. sings.

Consequence: $\mathrm{char}(K) = 0$:

$$\chi(\mathcal{O}_I \cdot \mathcal{O}^J) = \begin{cases} 1 & \text{if } J \leq I \\ 0 & \text{if } J \nleq I. \end{cases}$$