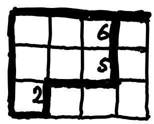
Cohomology of Grassmannian

$X = Gr(m, n)$.

$\binom{n}{m} \longleftrightarrow \{ \text{partitions } \lambda \subseteq (n-m)^m \}$

$I = \{ i_1 < \dots < i_m \} \longleftrightarrow \lambda = (i_m - m \geq \dots \geq i_1 - 1)$

$|I| = |\lambda|$.

$I = \{2, 5, 6\}$  $\lambda = (3, 3, 1)$.

Tautological subbundle: $\mathcal{S} \subseteq X \times K^n$.

<u>Thm</u> $S_\lambda(\mathcal{S}^\vee) = \begin{cases} [X^\lambda] & \text{if } \lambda \subseteq (n-m)^m \\ 0 & \text{else.} \end{cases}$

<u>Cor</u> $[X^\lambda] \cdot [X^\mu] = \sum_{\nu \subseteq (n-m)^m} c^\nu_{\lambda\mu} [X^\nu] \in A(X)$

given by LR rule.

<u>Proof</u> $\Lambda \longrightarrow A(X)$, $f \mapsto f(\mathcal{S}^\vee)$ ring hom.

Example

$X^{\square} = \{ V \in X \mid V \cap \langle e_{m+1}, \dots, e_n \rangle \neq 0 \}$ divisor.

$X^{\square\square} = \{ V \in X \mid V \cap \langle e_{m+2}, \dots, e_n \rangle \neq 0 \}$ codim. 2.

$X^{\substack{\square \\ \square}} = \{ V \in X \mid \dim(V \cap \langle e_m, \dots, e_n \rangle) \geq 2 \}$ codim. 2.

$[X^{\square}]^2 = [g.X^{\square} \cap X^{\square}]$ for general $g \in G$.

and $g.X^{\square} \cap X^{\square}$ irred. Richardson variety.

$E' = \langle e_{m+1}, \dots, e_n \rangle$, $E'' = \langle e_m, e_{m+2}, \dots, e_n \rangle$.

Choose $g \in G$ such that $g.E' = E''$.

$V \in g.X^{\square} \cap X^{\square}$

$\Updownarrow$

$V \cap E' \neq 0$ and $V \cap E'' \neq 0$

$\Updownarrow$

$V \cap (E' \cap E'') \neq 0$ or $\dim(V \cap (E' + E'')) \geq 2$.

$\therefore \; g.X^{\square} \cap X^{\square} = X^{\square\square} \cup X^{\substack{\square \\ \square}}$

<u>Check:</u> $g.X^{\square} \cap X^{\square}$ reduced.

$X^{\square\square} \cap X^{\substack{\square \\ \square}} = X^{\boxplus}$ reduced.

$[X^{\square}]^2 = [X^{\square\square}] + [X^{\substack{\square \\ \square}}] \in A(X)$.

$$0 \longrightarrow \mathcal{O}_{X^{\boxplus}} \longrightarrow \mathcal{O}_{X^{\square\square}} \oplus \mathcal{O}_{X^{\substack{\square \\ \square}}} \longrightarrow \mathcal{O}_{g.X^{\square} \cap X^{\square}} \longrightarrow 0$$

$\mathcal{O}^{\square} \cdot \mathcal{O}^{\square} = \mathcal{O}^{\square\square} + \mathcal{O}^{\substack{\square \\ \square}} - \mathcal{O}^{\boxplus} \in K(X)$.

Tools for K-theoretic intersection theory

Def $f : X \longrightarrow Y$ morphism of schemes.
f is cohomologically trivial if

$$f_* \mathcal{O}_X = \mathcal{O}_Y , \qquad R^i f_* \mathcal{O}_X = 0 \text{ for } i > 0.$$

Implies: $f_* [\mathcal{O}_X] = [\mathcal{O}_Y] \in K_0(Y)$.

X is cohom. triv. if $X \longrightarrow \mathrm{Spec}(K)$ cohom. triv:

$$H^0(X, \mathcal{O}_X) = K , \quad H^i(X, \mathcal{O}_X) = 0 \text{ for } i > 0.$$

Obs: $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphisms.
Assume f cohom. trivial. Then
g cohom. triv $\Leftrightarrow$ gf cohom. triv.

Proof Grothendieck spectral seq.:

$$R^i g_* (R^j f_* \mathcal{O}_X) \Rightarrow R^{i+j} (gf)_* \mathcal{O}_X$$

$$R^i g_* \mathcal{O}_Y = R^i (gf)_* \mathcal{O}_X .$$

□

Desingularization

X irred. variety

A resolution of singularities is a proper birational morphism

$$\pi : \tilde{X} \longrightarrow X$$

with $\tilde{X}$ non-singular.

Hironaka : $\mathrm{Char}(K) = 0$.

- Any irred. variety has resol. of sings.
- X and Y non-singular varieties.
  $f : X \xrightarrow{\approx} Y$ birational projective morph.
  Then f is cohom. trivial.

Assume $K = \bar{K}$, $\mathrm{char}(K) = 0$.

Def X variety,

$\pi : \tilde{X} \longrightarrow X$ resol. of sings.

X has rational singularities

if π cohom. trivial.

Independent of $\tilde{X}$:

$\hat{\pi} : \hat{X} \longrightarrow X$ other desingularization.

$\tilde{X} \times_X \hat{X}$ contains unique irred. comp. Y dominating X.

Choose desingularization $\tilde{Y} \xrightarrow{\approx} Y$

$\exists$ commutative diagram

$$\begin{tikzcd}
\tilde{Y} \arrow[r, "\approx", "\text{birat.}"'] \arrow[d, "\approx"', "\text{birat.}"] & \hat{X} \arrow[d, "\hat{\pi}"] \\
\tilde{X} \arrow[r, "\pi"'] & X
\end{tikzcd}$$

$\tilde{Y}$ non-sing.

$\hat{\pi}$ cohom. triv. $\Leftrightarrow$ π cohom. trivial.

Fact:

$\pi : \tilde{X} \longrightarrow X$ proper birational map of irred. varieties.

- X normal $\Rightarrow \pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$
- $\tilde{X}$ normal and $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$
  $\Rightarrow X$ normal.

Proof WLOG $X = \mathrm{Spec}(R)$ affine.

π proper $\Rightarrow \pi_* \mathcal{O}_{\tilde{X}}$ coherent $\mathcal{O}_X$-mod.

$R \underset{\text{finite}}{\subseteq} \mathcal{O}_{\tilde{X}}(\tilde{X}) \subseteq K(R)$.

R normal $\Rightarrow \mathcal{O}_{\tilde{X}}(\tilde{X}) \subseteq \bar{R} = R$.

$\tilde{X}$ normal $\Rightarrow \mathcal{O}_{\tilde{X}}(\tilde{X}) = \bigcap_{p \in \tilde{X}} \mathcal{O}_{p,\tilde{X}}$ normal.

$\square$

$\therefore$ rational singularities $\Rightarrow$ normal.

Fact Rational singularities $\Rightarrow$ Cohen Macaulay.

<u>Fact</u> (Kollár + translation) $k = \mathbb{C}$.

$f: X \longrightarrow Y$ surjective morphism between projective varieties $/\mathbb{C}$ with rat. sings.

Then f is cohom. trivial $\Leftrightarrow$ the general fibers of f are cohom. triv:

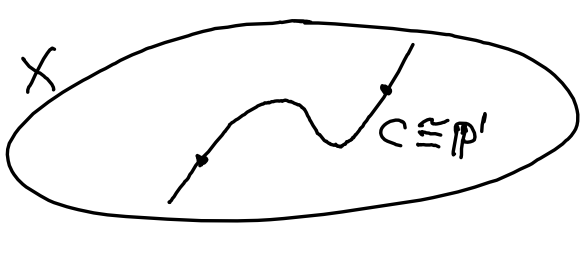
$\exists$ dense open $U \subseteq Y$:
$\forall y \in U$ closed: $f^{-1}(y)$ is cohom. triv.

<u>Fact</u> (Debarre) $k = \bar{k}$ uncountable, $\mathrm{char}(k) = 0$.

X non-singular projective rationally connected variety.

Then X is cohom. trivial.

<u>Rationally connected</u>: A general pair of points in X is connected by a rational curve $C \subseteq X$. E.g. X rational.



Fact (Graber, Harris, Starr)

$f: X \longrightarrow Y$ dominant morphism of complete varieties $/\mathbb{C}$.

If Y and the general fibers of f are rationally connected, then X is rationally connected.

Example

$E \subseteq \mathbb{P}^2$ elliptic curve $/\mathbb{C}$.

$X \subseteq \mathbb{P}^3$ cone over E, $x_0 = $ vertex.

$\tilde{X} = Bl_{x_0}(E)$.

Rational map $f: X \dashrightarrow E$

extends to morphism $\tilde{f}: \tilde{X} \longrightarrow E$.

$\tilde{f}^{-1}(p) \cong \mathbb{P}^1 \quad \forall \; p \in E$.

$\Rightarrow \tilde{f}$ cohom. trivial.

$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \chi(E, \mathcal{O}_E) = 0$.

$p_a(\tilde{X}) = (-1)^2 (\chi(\mathcal{O}_{\tilde{X}}) - 1) = -1$

Equivariant cohomology

G Lie group, $G \circlearrowright X$.

__Choose__: contractible space EG
with free (right) G-action

__Borel construction__

$$X_G = EG \times^G X = (EG \times X)/G = \{[e,x]\}$$

$$g.(e,x) = (eg^{-1}, g.x)$$

$$[eg, x] = [e, g.x]$$

__Def__ $H_G^*(X) = H^*(X_G)$.

__Classifying space__: $BG = EG/G$.

$$\Lambda_G = H_G^*(pt) = H^*(BG)$$

$EG \times X \longrightarrow EG$ G-equiv.

$p : X_G \longrightarrow BG$ X-bundle.

$p^* : \Lambda_G \longrightarrow H_G^*(X)$ ring hom.

$\therefore$ $H_G^*(X)$ is a Λ_G-algebra.
(if Λ_G commutative.)

Example

$G = \mathbb{C}^*$.

$EG = \mathbb{C}^\infty - \{0\}$ is contractible!

$e \in EG$, $g \in \mathbb{C}^*$.

$eg = (e_1 g, e_2 g, e_3 g, \dots)$

$BG = EG/G = \mathbb{P}^\infty$.

$\Lambda_{\mathbb{C}^*} = H^*(BG) = \mathbb{Z}[t]$.

Convention: $t = c_1(\mathcal{O}(-1))$.

Example

$m \in \mathbb{Z}$.

$\mathbb{C}_m = \mathbb{C}$ with $\mathbb{C}^*$-action $g \cdot z = g^m z$.

$$(\mathbb{C}_m)_G = EG \times^G \mathbb{C}_m \longrightarrow EG/G = \mathbb{P}^\infty$$

$= $ line bundle. $\mathcal{O}(m)$ or $\mathcal{O}(-m)$?

$EG \dashrightarrow EG \times \mathbb{C}_m$, $e \mapsto (e, e_i^{-m})$.

G-equivariant: $eg \mapsto (eg, (g e_i)^{-m}) = g^{-1} \cdot (e, e_i^{-m})$

Gives section: $\sigma_i : D_+(e_i) \longrightarrow (\mathbb{C}_m)_G$.

$\sigma_i / \sigma_j = (e_i / e_j)^{-m} \Rightarrow (\mathbb{C}_m)_{\mathbb{C}^*} = \mathcal{O}_{\mathbb{P}^\infty}(-m)$.