Divisors

X irred. variety. $\dim(X) = n$.

Cartier divisor:

Global section D of $\mathcal{K}^*/\mathcal{O}^*$.

$\mathcal{K} : U \mapsto R(U) = R(X)$ constant sheaf.

D rep. by $\{(U_\alpha, f_\alpha)\}$:

$$f_\alpha \in R(U_\alpha)^* = R(X)^*.$$

$$f_\alpha / f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$$

nowhere vanishing reg. fcn on $U_\alpha \cap U_\beta$.

Principal Cartier div:

Image of global section of $\mathcal{K}^*$.

$f \in R(X)^*$. $\quad \operatorname{div}(f) = \{(X, f)\}$.

$V \subseteq X$ prime divisor:

$$\operatorname{ord}_V(D) = \operatorname{ord}_V(f_\alpha), \quad V \cap U_\alpha \neq \emptyset.$$

Weil divisor: $[D] = \sum_V \operatorname{ord}_V(D)\,[V]$.

$[\operatorname{div}(f)] = \sum \operatorname{ord}_V(f)\,[V]$ principal Weil div.

Effective Cartier: $D = \{(U_\alpha, f_\alpha)\}$

$f_\alpha \in \mathcal{O}_X(U_\alpha)$, $\forall \alpha$

Then $D \subseteq X$ closed subscheme.

$[D]$ = fund. cycle.

$\mathrm{Div}(X)$ = group of Cartier divisors.

$Z_{n-1}(X)$ = group of Weil divisors.

$\mathrm{Pic}(X) = \mathrm{Div}(X) / \{\text{principal divs}\}$

$$\begin{tikzcd}
R(X)^* \arrow[r] \arrow[d, equal] & \mathrm{Div}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
R(X)^* \arrow[r] & Z_{n-1}(X) \arrow[r] & A_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

$D \in \mathrm{Div}(X)$.

$$|D| = \mathrm{Supp}(D) = \bigcup_{\mathrm{ord}_V(D) \neq 0} V \subseteq X \text{ closed subset.}$$

Line bundle: $\mathcal{O}_X(D) \subseteq \mathcal{K}$

Subsheaf gen. by f_α^{-1} on U_α.

Rational section: $1 \in \Gamma(X, \mathcal{K})$.

$$s_D = 1|_{X-|D|} \in \Gamma(X-|D|, \mathcal{O}_X(D))$$

$$s_D : \mathcal{O}_{X-|D|} \xrightarrow{\cong} \mathcal{O}_X(D)|_{X-|D|}.$$

Note: $f_\alpha = s_D / \text{generator of } \mathcal{O}_X(D)$ on U_α.

$$\Rightarrow \operatorname{ord}_V(s_D) = \operatorname{ord}_V(D).$$

Note: $\operatorname{div}(s_D) = D \in \operatorname{Div}(X)$.

Pseudo-Divisors

Idea: $0 \neq s \in \Gamma(X, L)$ section of line bundle.

$c_1(L) = [\mathrm{div}(s)] \in A_{n-1}(X)$, $\quad n = \dim X$.

$c_1(L) \neq 0 \Rightarrow Z(s) \neq \emptyset$.

Want:
$$\begin{array}{ccc} 0 \neq [\mathrm{div}(s)] & \in & A_{n-1}(Z(s)) \\ \downarrow & & \downarrow \\ [\mathrm{div}(s)] & \in & A_{n-1}(X) \end{array}$$

– more direct reason for $Z(s) \neq \emptyset$.

X scheme.

Def Pseudo-divisor on X:

(L, Z, s):

L line bundle on X.

$Z \subseteq X$ closed subset.

$s \in \Gamma(X-Z, L)$: $\quad s : \mathcal{O}_{X-Z} \xrightarrow{\cong} L|_{X-Z}$.

$(L, Z, s) = (L', Z', s')$:

$Z = Z'$, $\quad L \underset{\text{on } X}{\xrightarrow{\cong}} L'$, $\quad \underset{\text{on } X-Z}{s \mapsto s'}$.

<u>Notation</u>: $D = (L, Z, s)$.

$\mathcal{O}_X(D) = L$

$|D| = Z$

$s_D = s$

A Cartier div. D' <u>represents</u> D if

$|D'| \subseteq Z$ and

$(\mathcal{O}_X(D'), Z, s_{D'}) = (L, Z, s)$.

Every pseudo-divisor D is represented.

If $|D| \neq X$, then D is rep. by <u>unique</u> Cartier divisor:

$$\mathrm{div}(s_D) \in \mathrm{Div}(X).$$

<u>Def</u> X variety, $\dim X = n$,

D pseudo-divisor on X.

$$[D] = \sum_V \mathrm{ord}_V(D') \in A_{n-1}(|D|)$$

where D' Cartier div. rep. D.

<u>Note</u>: $|D| \neq X \Rightarrow A_{n-1}(|D|) = Z_{n-1}(|D|) \subseteq Z_{n-1}(X)$.

Pull-back:

$f: X' \longrightarrow X$ morphism.

$D = (L, Z, s)$ pseudo-divisor on X.

$f^*D = (f^*L, f^{-1}(Z), f^*s)$ pseudo-div. on X'.

Intersections

D pseudo-divisor on X.

$j: V \subseteq X$ closed subvariety. $\dim V = k$.

j^*D pseudo-divisor on V

$|j^*D| = V \cap |D|$.

Def: $D \cdot [V] = [j^*D] \in A_{k-1}(|D| \cap V)$

$\alpha = \sum n_V [V] \in Z_k(X)$.

$$|\alpha| = \bigcup_{n_V \neq 0} V \subseteq X \text{ closed.}$$

Def: $D \cdot \alpha = \sum n_V \, D \cdot [V] \in A_{k-1}(|D| \cap |\alpha|)$.

Example

$X = \mathbb{P}^n = \mathrm{Proj}\; K[x_0, \dots, x_n]$.

$D = (\mathcal{O}_X(1),\; Z(x_n),\; x_n)$ pseudo-divisor.

$V = Z(x_0, x_1) \subseteq X$. $\quad V \cong \mathbb{P}^{n-2}$

$$D \cdot [V] = [\,(\mathcal{O}_V(1),\; V \cap Z(x_n),\; x_n)\,]$$
$$= [\,Z(x_0, x_1, x_n)\,]$$
$$\in A_{n-3}(Z(x_0, x_1, x_n))$$
$$= Z_{n-3}(Z(x_0, x_1, x_n))$$

$V' = Z(x_0, x_n)$.

$$D \cdot [V'] = [(\mathcal{O}_{V'}(1),\; V',\; 0)]$$

$\in A_{n-3}(V')$ generator.

– no cycle obtained from D.

Projection formula

D pseudo-divisor on X.

$f: X' \longrightarrow X$ proper.

$\alpha \in Z_k(X')$.

Then $f_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \in A_{k-1}(X)$.

Refinement:

$f^*D \cdot \alpha \in A_{k-1}(f^{-1}(|D|) \cap |\alpha|)$

$f': f^{-1}(|D|) \cap |\alpha| \longrightarrow |D| \cap f(|\alpha|)$

restricted map.

$f'_*(f^*D \cdot \alpha) \in A_{k-1}(|D| \cap f(|\alpha|))$

$f_*(\alpha) \in Z_k(f(|\alpha|))$

$D \cdot f_*(\alpha) \in A_{k-1}(|D| \cap f(|\alpha|))$

$f'_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \in A_{k-1}(|D| \cap f(|\alpha|))$

Thm X variety, $\dim X = n$,

D, D' Cartier divisors on X.

Then $D \cdot [D'] = D' \cdot [D] \in A_{n-2}(|D| \cap |D'|)$.

$$\sum \mathrm{ord}_{V'}(D') \, D \cdot [V'] = \sum_{V} \mathrm{ord}_V(D) \, D' \cdot [V]$$

Cor X scheme. D pseudo-divisor.

$\alpha \in \mathrm{Rat}_k(X) \subseteq Z_k(X)$.

Then $D \cdot \alpha = 0 \in A_{k-1}(|D|)$.

Proof

WLOG: $\alpha = [\mathrm{div}(f)]$, $f \in R(V)^*$

$V \subseteq X$ subvariety. $\dim V = k+1$.

D' Cartier divisor representing D.

$$\begin{aligned} D' \cdot \alpha &= D' \cdot [\mathrm{div}(f)] \\ &= \mathrm{div}(f) \cdot [D'] \\ &= 0 \in A_{k-1}(|D'|). \end{aligned}$$

□

X scheme, $Y \subseteq X$ closed subscheme.
D pseudo-divisor on X.

$$Z_k(Y) \longrightarrow A_{k-1}(|D| \cap Y)$$
$$\alpha \longmapsto D \cdot \alpha$$

Cor: $\alpha \sim 0 \Rightarrow D \cdot \alpha = 0$.

<u>Intersection with D:</u>

$$A_k(Y) \longrightarrow A_{k-1}(|D| \cap Y)$$
$$\alpha \longmapsto D \cdot \alpha$$

<u>Cor</u> X scheme, D, D' pseudo-divisors, $\alpha \in A_k(X)$. Then
$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha) \in A_{k-2}(|D| \cap |D'| \cap |\alpha|).$$

<u>Proof</u>
WLOG: $\alpha = [V]$.
Replace D, D' with $D|_V$, $D'|_V$.

$$D \cdot [D'] = D' \cdot [D] \in A_{k-2}(|D| \cap |D'|).$$

$\square$

X scheme, $D_1, D_2, \ldots, D_n$ pseudo-divisors.

$\alpha \in Z_k(X)$.

$$D_1 \cdot D_2 \cdot \cdots \cdot D_n \cdot \alpha := D_1 \cdot (\cdots (D_{n-1} \cdot (D_n \cdot \alpha)) \cdots)$$
$$\in A_{k-n}(|D_1| \cap \cdots \cap |D_n| \cap |\alpha|).$$

– independent of order.

$P \in \mathbb{Z}[T_1, \ldots, T_n]$ homogeneous deg. d.

$$P(D_1, \ldots, D_n) \cdot \alpha \in A_{k-d}(|D_1| \cap \cdots \cap |D_n| \cap |\alpha|).$$

<u>Intersection number</u>

$\alpha \in Z_n(X)$, $D_1, \ldots, D_n$ pseudo-divisors.

$Y = |D_1| \cap \cdots \cap |D_n| \cap |\alpha|$.

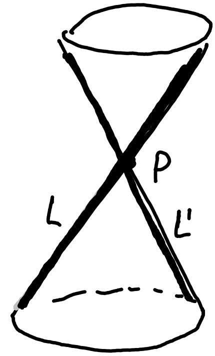
If Y complete:

$$(D_1 \cdot D_2 \cdots D_n \cdot \alpha)_X = \int_Y D_1 \cdot D_2 \cdots D_n \cdot \alpha$$

Example

$\mathbb{P}^3 = \text{Proj } K[x, y, z, t]$.

$X = V(z^2 - xy)$.



$L = V(x, z)$.

$L' = V(y, z)$.

$P = (0:0:0:1)$

$D = \{x = 0\} \subseteq X$ Cartier divisor.

$[D] = 2[L]$. $\quad \mathcal{O}_{L,X} = \left(K[x,y,z]/\langle z^2 - xy\rangle\right)_{\langle x, z\rangle}$

$\mathfrak{m}_{L,X} = \langle z \rangle$

$x = y^{-1} z^2 \quad \text{ord}_L(x) = 2$.

$D \cdot [L'] = [P]$. (x reduced equation for $P \in L'$.)

Assume $\exists$ Cartier div. D' on X

such that $[D'] = [L'] \in A_1(X)$.

$[P] = D \cdot [L'] = D \cdot [D'] = D' \cdot [D] = 2 D' \cdot [L]$

$1 = \int [P] = 2 \int D' \cdot [L]$.