Chern class of line bundle.

X scheme, L line bundle on X.

Pseudo-divisor: $D = (L, X, 0)$.

$$c_1(L) : A_k(X) \longrightarrow A_{k-1}(X)$$
$$\alpha \longmapsto c_1(L) \cap \alpha = D \cdot \alpha.$$

Projection formula

$f : X' \longrightarrow X$ proper, L line bundle on X,

$\alpha \in A_k(X')$.

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \in A_{k-1}(X).$$

Flat pullback

$f : X' \longrightarrow X$ flat of relative dim. n,

L line bundle on X, $\alpha \in A_k(X)$.

$$f^*(c_1(L) \cap \alpha) = c_1(f^*L) \cap f^*(\alpha) \in A_{k+n-1}(X').$$

Additivity

L, L' line bundles on X, $\alpha \in A_k(X)$.

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_2(L) \cap \alpha \in A_{k-1}(X)$$

Example

$$A_k(\mathbb{P}^n) \longrightarrow A_0(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$
$$\alpha \longmapsto c_1(\mathcal{O}(1))^k \cap \alpha \mapsto \int c_1(\mathcal{O}(1))^k \cap \alpha$$
$$[L^k] \longmapsto 1$$
$$\Rightarrow A_k(\mathbb{P}^n) = \mathbb{Z}.$$

Gysin maps for Divisors

X scheme, $i : D \subseteq X$ effective Cartier div.

Gysin homomorphism:

$$i^* : A_k(X) \longrightarrow A_{k-1}(D)$$
$$i^*(\alpha) = D \cdot \alpha.$$

Properties:

- $\alpha \in A_k(X)$: $\quad i_* i^*(\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha$
- $\alpha \in A_k(D)$: $\quad i^* i_*(\alpha) = c_1(N) \cap \alpha$

$N = i^* \mathcal{O}_X(D)$ normal bundle.

$$i^* i_* [V] = i^*[V] = D \cdot [V]$$
$$= i^* D \cdot [V] = c_1(N) \cap [V].$$

- L line bundle on X, $\alpha \in A_k(X)$.

$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*(\alpha) \in A_{k-2}(D)$.

$D' = (L, X, 0)$.

$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha) \in A_{k-2}(|D| \cap |\alpha|)$.

<u>Example</u>

X scheme, $p : L \longrightarrow X$ line bundle.

$p^* : A_k(X) \longrightarrow A_{k+1}(L)$.

$i : X \xrightarrow{\subseteq} L$ zero section.

$X \subseteq L$ effective Cartier.

$i^*(p^*(\alpha)) = \alpha$ for $\alpha \in A_k(X)$:

$$p^*([V]) = [p^{-1}(V)] \in A_{k+1}(L).$$

$$i^*[p^{-1}(V)] = X \cdot [p^{-1}(V)]$$

Restrict local equation of $X \subseteq L$ to $p^{-1}(V) \subseteq L$.

Result: $[X \cap p^{-1}(V)] = [V]$.

$\therefore$ $p^* : A_k(X) \xrightarrow{\cong} A_{k+1}(L)$ iso.

Example

$X \subseteq \mathbb{P}^n$ closed subscheme.

$X' \subseteq \mathbb{P}^{n+1}$ cone over X.

$\mathbb{P}^{n+1} \setminus \{0\} = \mathbb{P}^n \times \mathbb{A}^1 \supseteq X \times \mathbb{A}^1.$

$X' = X \times \mathbb{A}^1 \cup \{0\}.$

$A_0(X') = \mathbb{Z} = \mathbb{Z}[0].$

$A_k(X') = A_k(X \times \mathbb{A}^1) \cong A_{k-1}(X), \quad k > 0.$

Projective bundle

$\pi : E \longrightarrow X$ vector bundle of rank $e+1$.

$\mathbb{P}(E) \xrightarrow{p_E} X$ projective bundle.

$$\mathbb{P}(E) = \{ (x, \ell) \mid x \in X, \ \ell \subseteq E(x) \text{ 1-dim subspace} \}.$$

$$\begin{tikzcd}
p_E^* E \arrow[r] \arrow[d] & \mathbb{P}(E) \arrow[d, "p_E"] \\
E \arrow[r, "\pi"] & X
\end{tikzcd}$$

$$p_E^* E = \left\{ (x, \ell, v) \;\middle|\; \begin{array}{l} x \in X, \\ \ell \subseteq E(x) \text{ 1-dim}, \\ v \in E(x) \end{array} \right\}.$$

$\mathcal{O}_E(-1) \subseteq p_E^* E$ tautological subbundle.

$$\mathcal{O}_E(-1) = \{ (x, \ell, v) \in p_E^* E \mid v \in \ell \}.$$

$\mathcal{O}_E(1) = \mathcal{O}_E(-1)^\vee \longrightarrow \mathbb{P}(E)$ line bundle.

$\mathcal{E}$ sheaf of sections of E.

$S^\bullet = \mathrm{Sym}^\bullet \mathcal{E}^\vee$ sheaf of $\mathcal{O}_X$-algebras.

$E = \underline{\mathrm{Spec}}(S^\bullet) \xrightarrow{\pi} X$

$P(E) = \underline{\mathrm{Proj}}(S^\bullet) \xrightarrow{p_E} X$.

Segre classes of E

$E \xrightarrow{\pi} X$ v.b. of rank $e+1$.

$p : P(E) \longrightarrow X$ proper + flat, rel. dim e.

Def $s_i(E) : A_k(X) \longrightarrow A_{k-i}(X)$

$$s_i(E) \cap \alpha = p_*\left(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*(\alpha)\right)$$

Projection formula

$f : X' \longrightarrow X$ proper, $\alpha \in A_k(X')$.

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

Pf
$$\begin{tikzcd} P(f^*E) \arrow[r, "f'"] \arrow[d, "p'"'] & P(E) \arrow[d, "p"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

$f'^* \mathcal{O}_E(1) = \mathcal{O}_{f^*E}(1)$.

$$\begin{aligned} f_*(s_i(f^*E) \cap \alpha) &= f_* p'_*\left(c_1(\mathcal{O}_{f^*E}(1))^{e+i} \cap p'^*(\alpha)\right) \\ &= p_* f'_*\left(c_1(\mathcal{O}_{f^*E}(1))^{e+i} \cap p'^*(\alpha)\right) \\ &= p_*\left(c_1(\mathcal{O}_E(1))^{e+i} \cap f'_* p'^*(\alpha)\right) \\ &= p_*\left(c_1(\mathcal{O}_E(1))^{e+i} \cap p^* f_*(\alpha)\right) \\ &= s_i(E) \cap f_*(\alpha). \end{aligned}$$

□

Flat pullback:

$f : X' \longrightarrow X$ flat, $E \to X$, $\alpha \in A_k(X)$.

$$f^*(s_i(E) \cap \alpha) = s_i(f^*E) \cap f^*(\alpha).$$

Basic properties:

$s_i(E) \cap \alpha = 0$ for $i < 0$

$s_0(E) \cap \alpha = \alpha$.

Proof $\alpha = [V]$. $V \subseteq X$.

$f : V \longrightarrow X$ proper.

$s_i(E) \cap [V] = s_i(E) \cap f_*[V]$

$= f_*(s_i(f^*E) \cap [V])$

WLOG: $X = V$. $n = \dim(X)$.

$s_i(E) \cap [X] \in A_{n-i}(X) = 0$ for $i < 0$.

$s_0(E) \cap [X] = m [X] \in A_n(X)$. $m \in \mathbb{Z}$.

$U \subseteq X$ open, $E|_U \cong U \times \mathbb{A}^{e+1}$.

$s_0(E|_U) \cap [U] = m[U] \in A_n(U)$.

$P(E|_U) = \mathbb{P}^e \times U \xrightarrow{p} U$

$p_*(c_1(\mathcal{O}(1))^e \cap p^*[U]) = [U]$. $\boxed{m=1}$

$\square$

Cor $p^*: A_k(X) \longrightarrow A_{k+e}(P(E))$

split monomorphism.

Proof

$$A_k(X) \underset{}{\overset{p^*}{\rightleftarrows}} A_{k+e}(P(E))$$

$$p_*(c_1(\mathcal{O}_E(1))^e \cap \beta) \longleftarrow\!\shortmid \beta$$

$$\alpha \longmapsto p^*(\alpha)$$

$$\shortparallel$$

$$s_0(E) \cap \alpha \longleftarrow\!\shortmid$$

$\square$