Rational Equivalence on Bundles

$\pi : E \longrightarrow X$ vector bundle of rank r.

$p : \mathbb{P}(E) \longrightarrow X$ flat of rel. dim. $r-1$.

Thm

(a) $\pi^* : A_{k-r}(X) \xrightarrow{\cong} A_k(E) \quad \forall k \in \mathbb{N}.$

(b) $\Theta_E : A_*(X)^{\oplus r} \xrightarrow{\cong} A_*(\mathbb{P}(E))$

$$\Theta_E(\alpha_0, .., \alpha_{r-1}) = \sum_{i=0}^{r-1} c_1(\mathcal{O}_E(1))^i \cap p^*(\alpha_i)$$

Informally: $A_*(\mathbb{P}(E))$ free $A_*(X)$-module with basis $c_1(\mathcal{O}_E(1))^i$, $0 \le i \le r-1$.

$$A_k(\mathbb{P}(E)) = \bigoplus_{j=0}^{r-1} A_{k-j}(X)$$

Q: what is $c_1(\mathcal{O}_E(1))^p \cap p^*(\alpha)$, $p \geq r$?

$\mathcal{O}_E(-1) \subseteq p^*E$ subbundle.

$$0 \longrightarrow \mathcal{O}_{P(E)} \longrightarrow p^*E \otimes \mathcal{O}_E(1) \longrightarrow Q \longrightarrow 0$$

Whitney:

$$c_r(p^*E \otimes \mathcal{O}_E(1)) = 0 \cdot c_{r-1}(Q) = 0$$

$$0 = c_r(E \otimes \mathcal{O}_E(1)) \cap p^*(\alpha) =$$

$$= \sum_{i=0}^{r} \binom{r-i}{r-i} c_1(\mathcal{O}_E(1))^{r-i} c_i(p^*E) \cap p^*(\alpha)$$

$p \geq r$:

$$c_p(\mathcal{O}_E(1))^p \cap p^*(\alpha)$$

$$= -\sum_{i=1}^{r} c_1(\mathcal{O}_E(1))^{p-i} \cap p^*(c_i(E) \cap \alpha)$$

Gysin homomorphism

$\pi : E \longrightarrow X$ vector bundle, rank r.

$s : X \longrightarrow E$ section. $\pi s = id_X$.

$$\pi^* : A_{k-r}(X) \xrightarrow{\cong} A_k(E)$$

Def: $s^* = (\pi^*)^{-1} : A_k(E) \longrightarrow A_{k-r}(X)$.

Regular sequence

A ring (= commutative Noetherian)

$a_1, \dots, a_d \in A$.

Def $(a_1, \dots, a_d)$ is a regular sequence if

$\langle a_1, \dots, a_d \rangle \subsetneq A$ proper ideal and

$$A/\langle a_1, \dots, a_{i-1}\rangle \underset{\subseteq}{\xrightarrow{a_i \cdot}} A/\langle a_1, \dots, a_{i-1}\rangle$$

injective $\forall\ 1 \le i \le d$.

(a_i is non-zero divisor on $A/\langle a_1, \dots, a_{i-1}\rangle$.)

Fact: A local: $(a_1, \dots, a_d)$ regular

$\Leftrightarrow$ any reordering regular.

Depth

$(A, \mathfrak{m})$ local ring.

$\text{depth}(A) =$ max. length of regular sequence $(a_1, \dots, a_d)$.

Fact: $(a_1, \dots, a_d)$ any max. regular seq.
$\Rightarrow d = \text{depth}(A)$.

Fact: $\text{depth}(A) \leq \dim(A)$.

Def A is Cohen-Macaulay (CM) if $\text{depth}(A) = \dim(A)$.

Fact: Regular local $\Rightarrow$ Cohen-Macaulay.

Fact: Let $a_1, \dots, a_d \in \mathfrak{m}$.

$$\dim(A/\langle a_1, \dots, a_d \rangle) \geq \dim(A) - d.$$

If A CM:

$$\dim(A/\langle a_1, \dots, a_d \rangle) = \dim(A) - d$$

$\Leftrightarrow (a_1, \dots, a_d)$ regular sequence.

Normal cone

Y scheme, $X \subseteq Y$ closed subscheme.

$\mathcal{I} = \mathcal{I}(X) \subseteq \mathcal{O}_Y$ ideal sheaf.

Conormal sheaf:

$$\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \quad - \ \mathcal{O}_X\text{-module.}$$

$\mathcal{O}_X$-algebra: $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$.

Normal cone:

$$C_X Y = \operatorname{Spec}\Big(\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\Big) \longrightarrow X$$

$$\mathbb{P}(C_X Y) = \operatorname{Proj}\Big(\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\Big) \longrightarrow X$$

Pullback

$f : Y' \longrightarrow Y$ morphism,

$X \subseteq Y$ closed

$X' = f^{-1}(X) \subseteq Y'$.

$$\begin{tikzcd} C_{X'}Y' \arrow[r, hook] & C_XY \times_X X' \arrow[r] \arrow[d] & X' \arrow[r, hook] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ & C_XY \arrow[r] & X \arrow[r, hook] & Y \end{tikzcd}$$

$\mathcal{I} = \mathcal{I}(X) \subseteq \mathcal{O}_Y$

$\mathcal{I}' = \mathcal{I}(X') \subseteq \mathcal{O}_{Y'}$.

$f^*\mathcal{I} \twoheadrightarrow \mathcal{I}'$ on Y' $\Rightarrow$

$$g^*\Big(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\Big) \twoheadrightarrow \bigoplus_{n\geq 0} \mathcal{I}'^n/\mathcal{I}'^{n+1}$$

surjection of $\mathcal{O}_{X'}$-algebras

$\Rightarrow$ $C_XY \times_X X' \underset{\text{closed}}{\hookleftarrow} C_{X'}Y'$

<u>Def</u> $i : X \subseteq Y$ is <u>regular embedding</u> of codim. d if:

$\forall p \in X$ $\exists$ open affine $p \in U \subseteq Y$ s.t. $I(X) \subseteq \mathcal{O}_Y(U)$ generated by regular sequence of length d.

<u>Fact:</u> i reg. embedding of codim. $d \Rightarrow$ $\mathcal{I}/\mathcal{I}^2$ locally free $\mathcal{O}_X$-module of rank d, and

$$\mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2) \cong \bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

$\Rightarrow$ $C_X Y = \mathrm{Spec}\, \mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2) \longrightarrow X$ vector bundle of rank d:

$N_X Y = C_X Y$ <u>normal bundle</u>.

sheaf of sections: $(\mathcal{I}/\mathcal{I}^2)^\vee$.

__Example__

$X \subseteq Y$ effective Cartier divisor.

$\mathcal{I} = \mathcal{O}_Y(-X)$.

$\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes \mathcal{O}_X = \mathcal{O}_Y(-X)|_X$

$N_X Y = \mathcal{O}_Y(X)|_X$.

__Example__ Y non-singular variety.

$D_1, D_2 \subseteq Y$ effective Cartier divisors.

$X = D_1 \cap D_2$

Assume $\dim(X) = \dim(Y) - 2$.

Then $X \subseteq Y$ regular embedding.

$$\mathcal{O}_Y(-D_1) \oplus \mathcal{O}_Y(-D_2) \twoheadrightarrow \mathcal{I} = \mathcal{I}(X) \subseteq \mathcal{O}_Y.$$

$$\mathcal{O}_Y(-D_1)|_X \oplus \mathcal{O}_Y(-D_2)|_X \xrightarrow{\cong} \mathcal{I} \otimes \mathcal{O}_X$$

$$N_X Y = \left(\mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2)\right)|_X.$$

Intersection Product

$i : X \hookrightarrow Y$ regular embedding, codim. d.

V scheme of pure dim. k.

$f : V \longrightarrow Y$ morphism.

<u>Example</u>: $V \subseteq Y$ closed subvariety.

$$W = f^{-1}(V) := V \times_Y X$$

$$\begin{tikzcd} C \arrow[r, hook] & N \arrow[r, "\pi"'] \arrow[d] & W \arrow[l, bend right, "s"'] \arrow[r, hook, "j"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ & N_X Y \arrow[r] & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$$\boxed{\begin{array}{l} N = g^* N_X Y \\ C = C_W V \end{array}}$$

<u>Fact</u>: C has pure dim. k.

$\left(C \underset{\text{open}}{\subseteq} P(C \oplus 1) = \text{exceptional divisor in } Bl_X(Y \times \mathbb{A}^1). \right)$

<u>Def</u> $X \cdot V = i^! [V]$

$:= s^*([C]) = (\pi^*)^{-1}([C]) \in A_{k-d}(W)$

pullback by zero section of N.

Example

Y of pure dim., $i : X \hookrightarrow Y$ regular, $V = Y$.

$$\begin{tikzcd} C \arrow[r, equal] & N \arrow[r, "\pi"'] \arrow[d] & W \arrow[l, bend right, "s"'] \arrow[r, hook] \arrow[d, equal] & V \arrow[d, equal] \\ & N_X Y \arrow[r] & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$$i^![Y] = s^*[N] = [X].$$

Example

$X \subseteq Y$ effective Cartier divisor,

$V \subseteq Y$ subvariety.

$W = X \cap V \subseteq Y$.

$V \subseteq X$: $W = V$.

$$\begin{tikzcd} C = \{0\} \subseteq N \arrow[r, "\pi"'] \arrow[d] & W \arrow[l, bend right, "s"'] \arrow[r, equal] \arrow[d] & V \arrow[d] \\ N_X Y \arrow[r] & X \arrow[r, hook, "i"] & Y \end{tikzcd} \qquad N_X Y = \mathcal{O}_Y(X)|_X$$

$$i^![V] = s^*[0] = c_1(\mathcal{O}_Y(X)) \cap [V]$$
$$= X \cdot [V] \quad \text{(product with Cartier)}$$

$V \not\subseteq X$: $W \subseteq V$ Cartier,

$$C = \mathcal{O}_V(W)|_W = \mathcal{O}_Y(X)|_W.$$

$$\begin{tikzcd} C \arrow[r, equal] & N \arrow[r, "\pi"'] \arrow[d] & W \arrow[l, bend right, "s"'] \arrow[r, hook] \arrow[d, "g"] & V \arrow[d] \\ & N_X Y \arrow[r] & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$$i^![V] = s^*[N] = [W] = X \cdot [V].$$