Refined Gysin Homomorphisms

$i: X \hookrightarrow Y$ regular codim d.

$f: Y' \to Y$ morphism of schemes.

$X' = X \times_Y Y'$.

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

Def $\quad i^!: A_k(Y') \longrightarrow A_{k-d}(X')$

$$[V] \longmapsto X \cdot V$$

Notation: $\quad X \cdot_Y \alpha = i^!(\alpha)$, $\alpha \in A_k(Y')$.

$$i^* = i^! : A_k(Y) \longrightarrow A_{k-d}(X)$$

Gysin homomorphism.

Thm

$i : X \hookrightarrow Y$ regular codim. d.

$Y'' \xrightarrow{p} Y' \xrightarrow{f} Y$ morphisms.

$$\begin{tikzcd} X'' \arrow[r, "i''"] \arrow[d, "q"'] & Y'' \arrow[d, "p"] \\ X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

(a) Assume p proper, $\alpha \in A_k(Y'')$.

$$i^!(p_*(\alpha)) = q_*(i^!(\alpha)) \in A_{k-d}(X').$$

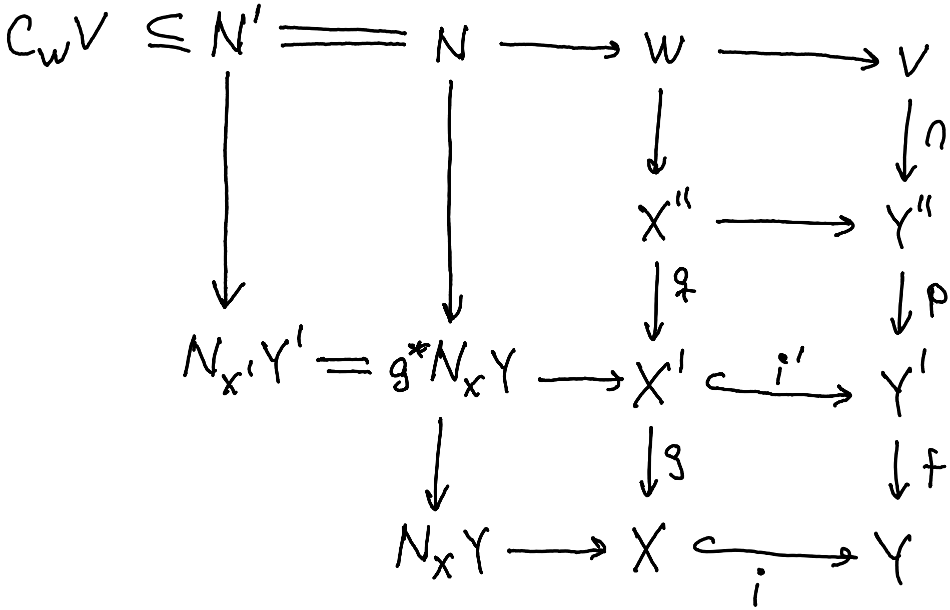
(b) Assume p flat of rel. dim. n, $\alpha \in A_k(Y')$.

$$i^!(p^*(\alpha)) = q^*(i^!(\alpha)) \in A_{k-d+n}(X'').$$

(c) Assume i' also regular embedding of codim. d, $\alpha \in A_k(Y'')$.

$$i'^!(\alpha) = i^!(\alpha) \in A_{k-d}(X'').$$

Part (c):



$$\begin{tikzcd}
C_W V \subseteq N' \arrow[r, equal] \arrow[dd] & N \arrow[r] \arrow[dd] & W \arrow[r] \arrow[d] & V \arrow[d, "\cap"] \\
 & & X'' \arrow[r] \arrow[d, "q"] & Y'' \arrow[d, "p"] \\
N_{X'} Y' \arrow[r, equal] & g^* N_X Y \arrow[r] \arrow[d] & X' \arrow[r, hook, "i'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\
 & N_X Y \arrow[r] & X \arrow[r, hook, "i"'] & Y
\end{tikzcd}$$

Example

$\pi : E \longrightarrow X$ vector bundle of rank r.

$s : X \longrightarrow E$ zero section.

Then $s^! = (\pi^*)^{-1} : A_*(E) \longrightarrow A_*(X)$.

Enough: $s^!(\pi^*([V])) = [V]$,

$V \subseteq X$ subvariety.

$\pi^*([V]) = [\pi^{-1}(V)]$.

$$\begin{tikzcd}
V \arrow[r, hook, "s'"] \arrow[d] \arrow[dr, phantom, "\square"] & \pi^{-1}(V) \arrow[d] \\
X \arrow[r, hook, "s"'] & E
\end{tikzcd}$$

s' and s both regular of codim. r.

$s^![\pi^{-1}(V)] = s'^![\pi^{-1}(V)] = [V]$.

Example

Let $[k] \in A_k(\mathbb{P}^n)$ be class of linear subspace.

$$\begin{aligned} \deg : A_k(\mathbb{P}^n) &\longrightarrow \mathbb{Z} \\ \alpha &\longmapsto \int c_1(\mathcal{O}(1))^{n-k} \cap \alpha \\ [k] &\longmapsto 1. \end{aligned}$$

Diagonal embedding is regular, codim. $(r-1)n$.

$\delta : \mathbb{P}^n \longrightarrow \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (r copies).

<u>Claim</u>: $\delta^*([k_1] \times \cdots \times [k_r]) = [\ell]$,

$$\ell = k_1 + \cdots + k_r - (r-1)n.$$

Let $K_1, \ldots, K_r \subseteq \mathbb{P}^n$ be linear subspaces in general position, $\dim(K_i) = k_i$.

$L = \bigcap_{i=1}^{r} K_i$. $\dim(L) = \ell$. (L empty if $\ell < 0$.)

$$\begin{tikzcd} L \arrow[r, hook, "\delta'"] \arrow[d] & K_1 \times \cdots \times K_r \arrow[d] \\ \mathbb{P}^n \arrow[r, hook, "\delta"'] & \mathbb{P}^n \times \cdots \times \mathbb{P}^n \end{tikzcd}$$

δ, δ' both regular of codim. $(r-1)n$.

$$\delta^*([k_1] \times \cdots \times [k_r]) = \delta'^*([K_1 \times \cdots \times K_r]) = [L].$$

Bezout's Theorem

$V_1, \ldots, V_r \subseteq \mathbb{P}^n$ closed subvarieties,

V_i of pure dim. k_i.

$$\deg \delta^*([V_1 \times \cdots \times V_i]) = \prod_{i=1}^{r} \deg(V_i).$$

Example

$$\mathbb{P}^5 = \operatorname{Proj} k[x_{ij},\ 1 \le i \le 2,\ 1 \le j \le 3].$$

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

$$\Omega = \{ x \in \mathbb{P}^5 \mid \operatorname{rank}(x) \le 1 \}.$$

Find $[\Omega] \in A_3(\mathbb{P}^5)$.

$$I(\Omega) = \langle f_{12}, f_{23}, f_{31} \rangle, \qquad f_{ij} = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix}.$$

Generate reduced ideal:

$D_+(x_{11})$: $\quad f_{12} = x_{22} - x_{21}x_{12}$

$$f_{23} = x_{12}x_{23} - x_{22}x_{13}$$

$$f_{31} = x_{13}x_{21} - x_{23}$$

$$\mathcal{O}(D_+(x_{11})) / \langle f_{12}, f_{23}, f_{31} \rangle \cong k[x_{21}, x_{12}, x_{13}].$$

Set $Z = V(f_{12}, f_{23}) \subseteq \mathbb{P}^5$.

$Z = \Omega \cup \Omega'$, $\quad \Omega' = V(x_{12}, x_{22})$.

$\supseteq$ clear.

$\subseteq$: Let $x \in Z \setminus \Omega'$. $\quad x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$

Then middle column is $\neq 0$, and first, third columns are multiples of middle column.

$\Rightarrow x \in \Omega$.

$[Z] = [\Omega] + [\Omega']$:

$$x_{13} f_{12} + x_{11} f_{23} + x_{12} f_{31} = 0$$
$$x_{23} f_{12} + x_{21} f_{23} + x_{22} f_{31} = 0$$

$$Z \cap D_+(x_{12}) = \Omega \cap D_+(x_{12}) \neq \emptyset$$
$$Z \cap D_+(f_{31}) = \Omega \cap D_+(f_{31}) \neq \emptyset.$$

$$\left.\begin{array}{l} \mathcal{O}_{\Omega,Z} = \mathcal{O}_{\Omega,\Omega} = R(\Omega) \\ \Omega_{\Omega',Z} = \mathcal{O}_{\Omega',\Omega'} = R(\Omega') \end{array}\right\} \text{fields.}$$

Bezout $\Rightarrow$

$$[Z] = Z(f_{12}) \cdot Z(f_{23}) = 4H^2$$

$$[\Omega'] = Z(x_{12}) \cdot Z(x_{22}) = H^2$$

$$[\Omega] = [Z] - [\Omega'] = 3H^2 \in A_3(\mathbb{P}^5).$$

Bivariant Chow groups

$f: X \longrightarrow Y$ morphism of schemes.

A <u>bivariant class</u> $c \in A^p(X \xrightarrow{f} Y)$ is a collection of homomorphisms

$$c_g^{(k)}: A_k(Y') \longrightarrow A_{k-p}(X \times_Y Y')$$

for all morphisms $g: Y' \longrightarrow Y$, all $k \in \mathbb{N}$,

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

that are compatible with proper push-forward, flat pullback, intersection products, i.e. must satisfy:

(C_1) Given $Y'' \xrightarrow{h} Y' \xrightarrow{g} Y$, h proper, $\alpha \in A_k(Y'')$:

$$c_g^{(k)}(h_*(\alpha)) = h'_*(c_{gh}^{(k)}(\alpha)) \in A_{k-p}(X').$$

$$\begin{tikzcd} X'' \arrow[r, "f''"'] \arrow[d, "h'"'] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "f'"'] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(C_2) Given $Y'' \xrightarrow{h} Y' \xrightarrow{g} Y$,

h flat of rel. dim. n,

$\alpha \in A_k(Y')$:

$$c_{gh}^{(k+n)}\left(h^*(\alpha)\right) = h'^*\left(c_g^{(k)}(\alpha)\right) \in A_{k+n-p}(X'').$$

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d, "h'"] & Y'' \arrow[d, "h"] \\
X' \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

(C_3) Given $Y' \xrightarrow{g} Y$, $Y' \xrightarrow{h} Z'$, $Z'' \xhookrightarrow{i} Z'$,

i regular embedding codim d,

$\alpha \in A_k(Y')$:

$$c_{gi'}^{(k-d)}\left(i^!(\alpha)\right) = i^!\left(c_g^{(k)}(\alpha)\right) \in A_{k-p-d}(X'').$$

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d, "i'"] & Z'' \arrow[d, hook, "i"] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r, "h"'] \arrow[d, "g"] & Z' \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$