# Example

$f: X \longrightarrow S = \mathrm{Spec}(K)$ structure morphism.

$$A^{-p}(X \longrightarrow S) \xrightarrow{\cong} A_p(X)$$
$$c \longmapsto c([S]) = c^{(0)}_{1_S}([S])$$

$$c = c^{(0)}_{1_S} : A_0(S) \longrightarrow A_p(X) \qquad \begin{array}{ccc} X & \longrightarrow & S \\ \| & & \| \\ X & \longrightarrow & S \end{array}$$

Inverse map:

Given $\beta \in A_p(X)$, def. $c_\beta \in A^{-p}(X \to S)$:

Given $g: Y' \longrightarrow S$,

$$\begin{array}{ccc} X \times Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow g \\ X & \longrightarrow & S \end{array}$$

$$(c_\beta)^{(k)}_g : A_k(Y') \longrightarrow A_{k+p}(X \times Y')$$
$$\alpha \longmapsto \beta \times \alpha$$

<u>Note</u>: $\beta \mapsto c_\beta \mapsto c_\beta([S]) = \beta \times [S] = \beta$.

<u>Show</u>: $c^{(k)}_g(\alpha) = c([S]) \times \alpha$

$\forall \alpha \in A_k(Y')$

$\forall c \in A^{-p}(X \to S)$

<u>WLOG:</u> $\alpha = [V]$, $V \subseteq Y'$ subvariety.

$$\begin{tikzcd}
X \times V \arrow[r] \arrow[d, "i'"] & V \arrow[d, "i"] \\
X \times Y' \arrow[r] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r] & S
\end{tikzcd}$$

$$\begin{aligned}
c(\alpha) &= c(i_*[V]) \\
&= i'_*(c([V])) \\
&= i'_*(c((gi)^*[S])) \\
&= i'_*(g'i')^*(c([S])) \\
&= i'_*(c([S]) \times [V]) \\
&= c([S]) \times [V] \\
&= c([S]) \times \alpha
\end{aligned}$$

## Example

1) $f: X \longrightarrow Y$ flat of rel. dim. $n$.

<u>orientation</u>: $[f] \in A^{-n}(X \xrightarrow{f} Y)$:

Given $g: Y' \longrightarrow Y$, $\alpha \in A_k(Y')$,

set $[f](\alpha) = f'^*(\alpha) \in A_{k+n}(X')$.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

2) $f: X \hookrightarrow Y$ regular embedding, codim. $d$.

<u>orientation</u>: $[f] \in A^d(X \xrightarrow{f} Y)$:

Given $g: Y' \longrightarrow Y$, $\alpha \in A_k(Y')$,

Set $[f](\alpha) = f^!(\alpha) \in A_{k-d}(X')$.

## Example

$E \longrightarrow X$ vector bundle of rank $r$, $i \in \mathbb{Z}$.

**Chern class** $c_i(E) \in A^i(X \xrightarrow{id} X)$:

Given $g : Y' \longrightarrow X$, $\alpha \in A_k(Y')$,

set $c_i(E)(\alpha) = c_i(g^*E) \cap \alpha \in A_{k-i}(Y')$.

$$\begin{tikzcd} Y' \arrow[r, equal] \arrow[d, "g"'] & Y' \arrow[d, "g"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

## Product

Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$:

$$A^p(X \xrightarrow{f} Y) \otimes A^q(Y \xrightarrow{g} Z) \longrightarrow A^{p+q}(X \xrightarrow{gf} Z)$$

$$c \otimes d \longmapsto cd$$

Given $Z' \longrightarrow Z$, $\alpha \in A_k(Z')$,

set $cd(\alpha) = c(d(\alpha)) \in A_{k-p-q}(X')$.

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

## Chow cohomology ring:

$$A^p(X) = A^p(X \xrightarrow{id} X).$$

$$A^*(X) = \bigoplus_p A^p(X).$$

Associative graded ring with $1 \in A^0(X)$.

## Pullback

Given morphisms $f : X \longrightarrow Y$, $g : Y_1 \longrightarrow Y$:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$g^* : A^p(X \longrightarrow Y) \longrightarrow A^p(X_1 \longrightarrow Y_1) :$$

$$g^*(c)(\alpha) = c(\alpha)$$

$c \in A^p(X \longrightarrow Y)$
$\alpha \in A_k(Y')$
$c(\alpha) \in A_{k-p}(X')$.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "h"] \\ X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note: $f^* : A^*(Y) \longrightarrow A^*(X)$
ring homomorphism.

Note: $A^*(X) \times A_*(X) \longrightarrow A_*(X)$
module structure.

$$A^p(X) \times A_k(X) = A^p(X \xrightarrow{id} X) \times A^{-k}(X \longrightarrow S).$$

# Push forward

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, $p$ proper.

$$f_* : A^p(X \xrightarrow{gf} Z) \longrightarrow A^p(Y \xrightarrow{g} Z)$$

Given $c \in A^p(X \xrightarrow{gf} Z)$,

$Z' \longrightarrow Z$, $\alpha \in A_k(Z')$:

$$(f_* c)(\alpha) = f'_*(c(\alpha)) \in A_{k-p}(Y').$$

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

## Projection Formula

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] & \\ X \arrow[r, "f"] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

$g$ proper.

$c \in A(X \xrightarrow{f} Y)$

$d \in A(Y' \xrightarrow{hg} Z)$

Then $c \cdot g_*(d) = g'_*(g^*(c) \cdot d)$

$\in A(X \xrightarrow{hf} Z)$.

## Special case

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "g"'] & Y \arrow[d, "g"] & \\ X \arrow[r, equal] & X \arrow[r] & \mathrm{Spec}(K) \end{tikzcd}$$

$g$ proper.

$c \in A^p(X)$, $\alpha \in A_k(Y)$.

$c \cdot g_*(\alpha) = g_*(g^*(c) \cdot \alpha)$.

$\therefore$ $g_* : A_*(Y) \longrightarrow A_*(X)$

homomorphism of $A^*(X)$-modules.

# Poincare duality

$X$ variety, $\dim(X) = n$.

$A^p(X) \longrightarrow A_{n-p}(X)$, $\quad c \mapsto c \cdot [X]$.

Next: $X$ non-singular variety
$\Rightarrow A^p(X) \cong A_{n-p}(X)$.

$f: Y \longrightarrow X$ morphism.

$\gamma_f : Y \hookrightarrow Y \times X$ graph of $f$.

Fact: $X$ non-singular variety of dim. $n$
$\Rightarrow \gamma_f$ regular of codim. $n$.

Assume $X$ non-singular, $\dim(X) = n$.
$V \subseteq X$ closed subvariety, $\dim(V) = m$.

<u>Def</u>: $c_V \in A^{n-m}(X)$:

Given $f : Y \to X$, $\alpha \in A_k(Y)$.

$$\begin{tikzcd} f^{-1}(V) \arrow[r] \arrow[d] & Y \times V \arrow[r, "pr_1"'] \arrow[d] & Y \\ Y \arrow[r, hook, "\gamma_f"'] & Y \times X & \end{tikzcd}$$

$$c_V(\alpha) = \gamma_f^!(pr_1^*(\alpha)) \in A_{k+m-n}(f^{-1}(V)).$$

This defines: $A_m(X) \longrightarrow A^{n-m}(X)$.
$[V] \longmapsto c_V$

<u>Note</u>:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \times V \arrow[r, "pr_1"'] \arrow[d] & X \\ X \arrow[r, "\delta"] & X \times X & \end{tikzcd}$$

$$\begin{aligned} c_V \cdot [X] &= \delta^!(pr_1^*([X])) \\ &= \delta^!([X \times V]) \\ &= [V] \end{aligned}$$

$\therefore$ $A_m(X) \longrightarrow A^{n-m}(X) \longrightarrow A_m(X)$ identity.
(opposite composition requires more work.)

Example

$$A^*(\mathbb{P}^n) = \mathbb{Z}[h]/\langle h^{n+1}\rangle.$$

$h = [H]$ class of hyperplane $H \subseteq \mathbb{P}^n$.

Example

$X$ scheme, $\pi: E \longrightarrow X$ vector bundle of rank $r$.

$p: P(E) \longrightarrow X$ flat of rel. dim. $r-1$.

$$A^*(X)[h] \longrightarrow A^*(P(E))$$
$$h \longmapsto c_1(\mathcal{O}_E(1))$$
$$c \longmapsto p^*(c) \quad \text{for } c \in A^*(X).$$

Recall: $p^*E \otimes \mathcal{O}_E(1)$ has non-vanishing section

$$\Rightarrow 0 = c_r(p^*E \otimes \mathcal{O}_E(1)) = \sum_{i=0}^{r} c_i(p^*E) \cdot c_1(\mathcal{O}_E(1))^{r-i}.$$

$$A^*(X)[h]/\langle h^r + c_1(E)h^{r-1} + \dots + c_r(E)\rangle \longrightarrow A^*(P(E)).$$

Thm This is an isomorphism of rings.