Some algebra review problems that will be needed in our course

1. (a) Use completion of squares to solve $2x^2 - 4x - 5 = 0$.

(b) Use completion of squares to find the center and radius of the circle $x^2+y^2-4x+6y-3=0$.

2. For the true-or-false questions, encircle *true* or *false*; then give a reason if the assertion is true or a counterexample if the assertion is false.

(a) *True* or *False:* If A and P are 2×2 matrices with P invertible and λ is an eigenvalue for A, then λ is an eigenvalue for $P^{-1}AP$.

(b) *True* or *False:* If A and P are 2×2 matrices with P invertible and $\mathbf{v}$ is an eigenvector for A, then $\mathbf{v}$ is an eigenvector for $P^{-1}AP$.

(c) *True* or *False:* If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbf{R}^n$, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution $\mathbf{x} = \mathbf{0}$.

(d) *True* or *False:* If A is a 3×2 matrix whose columns $\mathbf{u}, \mathbf{v}$ are mutually orthogonal, then $A^{\mathrm{T}}A$ is a diagonal matrix.

(e) *True* or *False:* If A and B are two $n \times n$ invertible matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

3. Suppose A is a 3×3 matrix and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are *nonzero vectors* in $\mathbf{R}^3$ such that

$$A\mathbf{u} = 2\mathbf{u}, \quad A\mathbf{v} = -2\mathbf{v}, \quad A\mathbf{w} = 0.$$

(a) Let $P = [\,\mathbf{u} \mid \mathbf{v} \mid \mathbf{w}\,]$ (the 3×3 matrix with columns $\mathbf{u}, \mathbf{v}, \mathbf{w}$). Find a 3×3 matrix D so that $AP = PD$. Prove that your answer is correct by calculating AP and PD separately.

(b) Let $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, where a, b, and c are scalars. Write the vectors $A\mathbf{x}$ and $A^2\mathbf{x}$ as linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

(c) Suppose a, b, and c are scalars such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0$. Prove that $a = 0$, $b = 0$, and $c = 0$. (Hint: Use **(b)** with $\mathbf{x} = 0$.)

(d) Prove that the matrix P in (a) is invertible and the matrix A is diagonalizable.

(e) Use **(a)** to find $\det A$ and the characteristic polynomial of A in factorized form.

4. Let W be the subspace of $\mathbf{R}^3$ spanned by the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find the orthogonal projection $\mathbf{w}$ of $\mathbf{v}$ onto W.

(b) Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $W^{\perp}$. Write down the equation satisfied by x_1, x_2, x_3. Use this to find a basis for $W^{\perp}$.

5. Let W be the subspace of $\mathbf{R}^4$ with basis vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ given below to obtain an orthogonal set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = \mathbf{u}_1$. Fill in the table on the left as you calculate. (Don't normalize $\mathbf{v}_2$ and $\mathbf{v}_3$.)

$\mathbf{v}_1 \cdot \mathbf{v}_1$	
$\mathbf{u}_2 \cdot \mathbf{v}_1$	
$\mathbf{v}_2 \cdot \mathbf{v}_2$	
$\mathbf{u}_3 \cdot \mathbf{v}_1$	
$\mathbf{u}_3 \cdot \mathbf{v}_2$	

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$\mathbf{v}_2 =$

$$\mathbf{u}_3 = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 =$$

(Problem 5 continues on next page)

(Continuation of Problem 5)

(b) Normalize the orthogonal basis for the subspace W from **(a)** to obtain an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for W.

(c) If $\mathbf{v} \in \mathbf{R}^4$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is any orthonormal basis for the subspace W, then the orthogonal projection of $\mathbf{v}$ onto W is the vector $\mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3$, where

$$c_1 = \underline{\hspace{3cm}} \qquad c_2 = \underline{\hspace{3cm}} \qquad c_3 = \underline{\hspace{3cm}}$$

(give a formula for the coefficients in terms of dot products).

(d) Take $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 0 \end{bmatrix}$ and use the formulas from part **(c)** and the orthonormal basis from part **(b)** to calculate the orthogonal projection of $\mathbf{v}$ onto W. Check your answer by calculating the vector $\mathbf{z} = \mathbf{v} - \mathbf{w}$ and showing that $\mathbf{z}$ is perpendicular to $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

6. Let

$$\mathbb{A} = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}.$$

Find the eigenvalues of $\mathbb{A}$ and prove that $\mathbb{A}$ is diagonalizable, *i.e.*, there is an invertible matrix $\mathbb{P}$ such that $\mathbb{P}^{-1}\mathbb{A}\mathbb{P}$ is diagonalizable. Furthermore, $\mathbb{P}$ can be chosen to be an orthogonal matrix, *i.e.*, a matrix whose columns are unit vectors and whose distinct columns are orthogonal to each other.

7. Let A be a 3×3 symmetric matrix with real entries, and have eigenvalues $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = 3$. Let $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be corresponding eigenvectors (normalized to have length one).

(a) Since A is symmetric and λ_1, λ_2, and λ_3 are all different, it follows that

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = \underline{\quad\quad}$$

(b) The 3×3 matrix $P = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3]$ satisfies

$P^{\mathrm{T}}P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ (fill in the entries of this 3×3 matrix).

(c) Let P be the matrix of normalized eigenvectors from **(b)**. Then $A = PDP^{\mathrm{T}}$, where

$D = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ (fill in the entries of this 3×3 matrix).

(d) The characteristic polynomial of A is __________________________.