

SOLUTIONS TO CLASSIFICATION PROBLEM (1.3,1-2)

(a). $x^2 - 4xy - 2y^2 + 6x + 12y + 21 = 0$

Matrix: $\begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix}$

Eigenvalues: $\lambda_1 = -3, \lambda_2 = 2$

Eigenvectors: $v_1 = [-1/\sqrt{5}, -2/\sqrt{5}], v_2 = [2/\sqrt{5}, -1/\sqrt{5}]$

Rotation matrix: $\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

Conic after rotation: $-3x^2 + 2y^2 - 6\sqrt{5}x + 21 = 0$

Translation: $(-\sqrt{5}, 0)$

Conic after translation: $\frac{1}{12}x^2 - \frac{1}{18}y^2 = 1$

Hyperbola

Center/Vertex: $[1, 2]$

Major axis: $-2x + y = 0$

Eccentricity: $e = \frac{1}{2}\sqrt{10}$

Foci: $[-\sqrt{6} + 1, -2\sqrt{6} + 2], [\sqrt{6} + 1, 2\sqrt{6} + 2]$

Directrices: $\frac{1}{2}x + y - \frac{5}{2} + \sqrt{6} = 0, \frac{1}{2}x + y - \frac{5}{2} - \sqrt{6} = 0$

(b). $5x^2 + 4xy + 5y^2 + 20x + 8y - 1 = 0$

Matrix: $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = 7$

Eigenvectors: $v_1 = [1/\sqrt{2}, -1/\sqrt{2}], v_2 = [1/\sqrt{2}, 1/\sqrt{2}]$

Rotation matrix: $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Conic after rotation: $3x^2 + 7y^2 + 6\sqrt{2}x + 14\sqrt{2}y - 1 = 0$

Translation: $(-\sqrt{2}, -\sqrt{2})$

Conic after translation: $\frac{1}{7}x^2 + \frac{1}{3}y^2 = 1$

Ellipse

Center/Vertex: $[-2, 0]$

Major axis: $x + y + 2 = 0$

Eccentricity: $e = 2/\sqrt{7}$

Foci: $[\sqrt{2} - 2, -\sqrt{2}], [-2 - \sqrt{2}, \sqrt{2}]$

Directrices: $-x + y - 2 + 7/\sqrt{2} = 0, -x + y - 2 - 7/\sqrt{2} = 0$

(c). $x^2 - 4xy + 4y^2 - 6x - 8y + 5 = 0$

Matrix: $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

Eigenvalues: $\lambda_1 = 0, \lambda_2 = 5$

Eigenvectors: $v_1 = [-2/\sqrt{5}, -1/\sqrt{5}], v_2 = [1/\sqrt{5}, -2/\sqrt{5}]$

Rotation matrix: $\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

Conic after rotation: $5y^2 + 4\sqrt{5}x + 2\sqrt{5}y + 5 = 0$

Translation: $(-1/\sqrt{5}, -1/\sqrt{5})$

Conic after translation: $y^2 + \frac{4}{\sqrt{5}}x = 0$

Parabola

Center/Vertex: $[1/5, 3/5]$

Major axis: $-x + 2y - 1 = 0$

Eccentricity: $e = 1$

Focus: $[3/5, 4/5]$

Directrix: $2x + y = 0$

(d). $21x^2 - 24xy + 31y^2 + 6x + 4y - 25 = 0$

Matrix: $\begin{bmatrix} 21 & -12 \\ -12 & 31 \end{bmatrix}$

Eigenvalues: $\lambda_1 = 13, \lambda_2 = 39$

Eigenvectors: $v_1 = [-3/\sqrt{13}, -2/\sqrt{13}], v_2 = [2/\sqrt{13}, -3/\sqrt{13}]$

Rotation matrix: $\begin{bmatrix} -3/\sqrt{13} & 2/\sqrt{13} \\ -2/\sqrt{13} & -3/\sqrt{13} \end{bmatrix}$

Conic after rotation: $13x^2 + 39y^2 - 2\sqrt{13}x - 25 = 0$

Translation: $(1/\sqrt{13}, 0)$

Conic after translation: $\frac{1}{2}x^2 + \frac{3}{2}y^2 = 1$

Ellipse

Center/Vertex: $[-3/13, -2/13]$

Major axis: $2x = 3y$

Eccentricity: $e = \frac{1}{3}\sqrt{6}$

(e). $3x^2 - 10xy + 3y^2 + 14x - 2y + 3 = 0$

Matrix: $\begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$

Eigenvalues: $\lambda_1 = -2, \lambda_2 = 8$

Eigenvectors: $v_1 = [-1/\sqrt{2}, -1/\sqrt{2}], v_2 = [1/\sqrt{2}, -1/\sqrt{2}]$

Rotation matrix: $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Conic after rotation: $-2x^2 + 8y^2 - 6\sqrt{2}x + 8\sqrt{2}y + 3 = 0$

Translation: $(-3/\sqrt{2}, -1/\sqrt{2})$

Conic after translation: $\frac{1}{4}x^2 - y^2 = 1$

Hyperbola

Center/Vertex: $[1, 2]$

Major axis: $-x + y - 1 = 0$

Eccentricity: $e = \frac{1}{2}\sqrt{5}$