Solutions to parctice problems

You are encouraged to draw pictures to illustrate all the problems!

Let $E \subset \mathbb{R}^2$ be the non-degenerate conic with equation $s = 0$, where
$s = x^2 + 2xy + y^2 + 2x - y - 3$.

4.2 (3). For the point $P_1 = (2, 1) \in \mathbb{R}^2$ we compute:
$s_1 = 2x + 2\frac{2y+1x}{2} + y + 2\frac{x+2}{2} - \frac{y+1}{2} - 3 = 4x + \frac{5}{2}y - \frac{3}{2}$
and $s_{11} = 9$.
The tangent pair to E from P_1 has equation $(s_1)^2 - s_{11}s = 0$.
We are told that one of the tangent lines has equation $y - 2x + 3 = 0$.
This is possible only if $y - 2x + 3$ is a factor of $(s_1)^2 - s_{11}s$.
A calculation gives $\dfrac{(s_1)^2 - s_{11}s}{y - 2x + 3} = \dfrac{1}{4}(-14x - 11y + 39)$.
So the other tangent line to E through P_1 has equation $14x + 11y - 39 = 0$.

4.2 (4). (a) For the point $P_2 = (0, \frac{3}{5})$ we compute $s_2 = \frac{8x}{5} + \frac{y}{10} - \frac{33}{10}$. The polar of P_2 wrt. E has equation $s_2 = 0$. The point $P_1 = (2, 1)$ satisfies this equation.
(b) The polar of P_1 wrt. E has equation $s_1 = 0$. The point P_2 satisfies this equation.

4.2 (5). Let $E \subset \mathbb{RP}^2$ be the projective conic with equation $s = 0$, where
$s = x^2 + y^2 - 2z^2 + 2xy - yz + 4zx$.
(a) Let $P_1 = [0, 1, -1]$, and notice that $P_1 \in E$.
We compute $s_1 = -x + \frac{3y}{2} + \frac{3z}{2}$.
The tangent to E at P_1 has equation $s_1 = 0$.
The point $P_2 = [3, 0, 2]$ satisfies this equation.
(b) For the point $P_2 = [3, 0, 2]$ we compute
$s_2 = 7x + 2y + 2z$ and $s_{22} = 25$.
The tangent pair from P_2 wrt. E has equation $(s_2)^2 - s_{22}s = 0$.
Since one of the tangents has equation $s_1 = 0$, s_1 is a divisor of $(s_2)^2 - s_{22}s$.
A calculation gives $\dfrac{(s_2)^2 - s_{22}s}{s_1} = -24x - 14y + 36z$.
The other tangent line to E through P_2 has equation $12x + 7y - 18z = 0$.
(c) The polar of P_2 wrt. E has equation $s_2 = 0$.

4.3 (1). Let $E \subset \mathbb{RP}^2$ be the non-degenerate conic through the points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, -1, 1]$, $[4, -1, -3]$.
Then E has an equation $Ax^2 + Bxy + Cy^2 + Fxz + Gyz + Hz^2 = 0$.
Since $[1, 0, 0] \in E$, we have $A = 0$.
Since $[0, 1, 0] \in E$, we have $C = 0$.
Since $[0, 0, 1] \in E$, we have $H = 0$.
So the equation of E is $Bxy + Fxz + Gyz = 0$.
Since $[1, -1, 1] \in E$, we have $-B + F - G = 0$. (*)
Since $[4, -1, -3] \in E$, we have $-4B - 12F + 3G = 0$. (**)
Add 3 times (*) to (**) to get $-7B - 9F = 0$.
If we take $B = 9$, then we get $F = -7$ and $G = F - B = -16$.
Equation for E: $9xy - 7xz - 16yz = 0$.

4.3 (2). Let $ABCD$ be a quadrilateral in $\mathbb{RP}^2$

Let $E \subset \mathbb{RP}^2$ be a projective conic through A, B, C, D.

Set $P = AB \cap CD$, $Q = AC \cap BD$, $R = AD \cap BC$.

We must prove that:

(a) The tangent to E at A, the tangent to E at B, and the line QR are concurrent.

(b) The line PQ is the polar of R wrt. E.

According to the three points theorem, there exists a projective transformation $t : \mathbb{RP}^2 \to \mathbb{RP}^2$ such that $t(A) = [1, 0, 0]$, $t(B) = [0, 1, 0]$, $t(C) = [0, 0, 1]$, and the image $t(E)$ has equation $xy + yz + zx = 0$. Since (a) and (b) are projective statements, it is enough to prove these statements for the images of E and the four points. So we may assume without loss of generality that:

E has equation $xy + yz + zx = 0$, and

$A = [1, 0, 0]$, $B = [0, 1, 0]$, $C = [0, 0, 1]$.

Since $D \in E$ and $D \neq A$, we may choose $d \in \mathbb{R}$ such that $D = [d^2 + d, d + 1, -d]$.

Now compute the following points and lines:

AB: $z = 0$.

CD: $x = dy$.

$P = AB \cap CD = [d, 1, 0]$.

AC: $y = 0$.

BD: $x = -(d + 1)z$.

$Q = AC \cap BD = [d + 1, 0, -1]$.

AD: $dy + (d + 1)z = 0$

BC: $x = 0$.

$R = AD \cap BC = [0, d + 1, -d]$.

QR: $x + dy + (d + 1)z = 0$.

Tangent to E at A: $y + z = 0$

Tangent to E at B: $x + z = 0$.

Intersection of tangents at A and B: $[1, 1, -1]$.

This point lies on QR, which proves (a).

Polar of R: $x - dy + (d + 1)z = 0$.

Both P and Q lie on this line, which proves (b).

4.3 (4). Let $E \subset \mathbb{RP}^2$ be the projective conic with equation $xy + yz + zx = 0$.

Let $A = [1, 0, 0]$, $B = [0, 1, 0]$, $C = [2, 2, -1]$, $D = [0, 0, 1]$.

Let $T \in E$ be any other point.

Set $B' = TB \cap AD$ and $C' = TC \cap AD$.

We must compute the cross ratio $(A\, B'\, C'\, D)$.

Then we may choose $t \in \mathbb{R}$ such that $T = [t^2 + t, t + 1, -t]$.

AD: $y = 0$.

TB: $x + (t + 1)z = 0$.

TC: $(t - 1)x + (t^2 - t)y + (2t^2 - 2)z = 0$.

$B' = TB \cap AD = [t + 1, 0, -1]$.

$C' = TC \cap AD = [2t^2 - 2, 0, 1 - t] = [2t + 2, 0, -1]$.

$(A\, D\, B'\, C') = \dfrac{-1/(t + 1)}{-1/(2t + 2)} = 2.$

$(A\, B'\, D\, C') = 1 - (A\, D\, B'\, C') = 1 - 2 = -1.$

$(A\, B'\, C'\, D) = (A\, B'\, D\, C')^{-1} = -1.$

4.3 (6). Let $E \subset \mathbb{RP}^2$ be the conic with equation $-2x^2 + 3xy + 3y^2 + 6xz + 6yz + 2z^2 = 0$.

Set $P = [1, -1, 1]$, $Q = [1, -2, 2]$, and $R = [1, -2, 1]$. These points lie on E.

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$.

Define $t : \mathbb{RP}^2 \to \mathbb{RP}^2$ by $t([v]) = [Av]$.

(c) Find the equation for $t(E)$.

$[x, y, z] \in t(E) \Leftrightarrow$

$t^{-1}([x, y, z]) \in E \Leftrightarrow$

$[x + y - z, -x - 2y + 2z, x + 2y - z] \in E \Leftrightarrow$

$xy + 3xz + 2yz = 0$.

This shows that $t(E)$ has equation $xy + 3xz + 2yz = 0$.

(d) Define $t' : \mathbb{RP}^2 \to \mathbb{RP}^2$ by $t'([x, y, z]) = [x/2, y/3, z]$.

Then $t'(t(E))$ has equation $xy + yz + zx = 0$.

The transformation $t' \circ t$ has associated matrix given by:

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ -1/3 & 0 & 1/3 \\ 0 & 1 & 1 \end{bmatrix}$$

4.3 (7). Let $E \subset \mathbb{RP}^2$ be a non-degenerate conic, let $P \in \mathbb{RP}^2$ be a point outside E, and let ℓ be a line through P that meets E in the points C and D. Let the tangent pair from P to E meet E in the points A and B, and let $Q = AB \cap \ell$. We must show that $(P\,Q\,C\,D) = -1$.

By using the three point theorem, we may assume that E has equation $xy+yz+zx = 0$ and that $A = [1, 0, 0]$, $B = [0, 1, 0]$, and $C = [0, 0, 1]$.

Then the tangent to E at A has equation $y + z = 0$.

And the tangent to E at B has equation $x + z = 0$.

Since P belongs to both of these tangents, we have $P = [1, 1, -1]$.

Since ℓ contains P and C, ℓ has equation $x = y$.

The line AB has equation $z = 0$.

We obtain $Q = AB \cap \ell = [1, 1, 0]$.

By solving the system of equations $\{xy + yz + zx = 0, x = y\}$, we find that $\ell \cap E = \{[0, 0, 1], [2, 2, -1]\}$. This implies that $D = [2, 2, -1]$.

It is easy to compute:

$(Q\,C\,P\,D) = \dfrac{-1/1}{-1/2} = 2$

We obtain

$(Q\,P\,C\,D) = 1 - 2 = -1$ and

$(P\,Q\,C\,D) = (Q\,P\,C\,D)^{-1} = -1$.