GOODIES 1

Problem 1. Let X be a Noetherian topological space.

(a) If an irreducible closed set Y is contained in a union $\bigcup X_i$ of finitely many closed sets X_i, then $Y \subset X_i$ for some i.

(b) X has finitely many components.

(c) X is the union of its components.

(d) X is not the union of any proper subset of its components.

Problem 2. Let X be any space with functions and $Y \subset \mathbb{A}^n$ an affine variety. Show that a function $f : X \to Y$ is a morphism if and only if each coordinate function $f_i : X \to k$ is regular for $1 \leq i \leq n$.

Problem 3. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \to k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Problem 4. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \to k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.

Problem 5. (a) $k[\mathbb{A}^n \smallsetminus \{0\}] = k[x_1, \ldots, x_n]$ for $n \geq 2$.

(b) $\mathbb{A}^n \smallsetminus \{0\}$ is not an affine variety for $n \geq 2$.

(c) Every global regular function on $\mathbb{P}^n$ is constant, i.e. $k[\mathbb{P}^n] = k$.

(d) $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Problem 6. Let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Problem 7. Let X be an affine variety and let $f \in k[X]$. Show that $D(f)$ is an affine variety with coordinate ring $k[D(f)] \cong k[X]_f$. Here $k[X]_f = S^{-1}k[X]$ is the localized ring defined by the multiplicatively closed subset $S = \{f^n \mid n \in \mathbb{N}\}$. (Hint: Show that $D(f)$ is isomorphic to a closed subset of $X \times \mathbb{A}^1$.)

Problem 8. Define the *homogenization* of a polynomial $f \in k[x_1, \ldots, x_n]$ to be $f^* = x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0)$. Equivalently, if we write $f = f_0 + f_1 + \cdots + f_d$, with f_i a form of degree i and $f_d \neq 0$, then $f^* = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + f_d \in k[x_0, x_1, \ldots, x_n]$.

Given any ideal $I \subset k[x_1, \ldots, x_n]$, let $I^* \subset k[x_0, x_1, \ldots, x_n]$ be the homogeneous ideal generated by $\{f^* \mid f \in I\}$.

(a) Find an example where $I = \langle h_1, \ldots, h_m \rangle$ and $I^* \neq \langle h_1^*, \ldots, h_m^* \rangle$.

(b) Let $X \subset \mathbb{A}^n$ be a closed subvariety. Identify $\mathbb{A}^n$ with $D_+(x_0) \subset \mathbb{P}^n$ and let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Show that $I(\overline{X}) = I(X)^* \subset k[x_0, \ldots, x_n]$.

Problem 9. Let R be a graded ring and let $f \in R$ be any homogeneous element. Then R_f is also a graded ring. Let $R_{(f)} \subset R_f$ be the subring of all elements of degree 0.

Problem 10. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Problem 11. Show that if R is a finitely generated reduced k-algebra then the space with functions Spec-m(R) is an affine variety.

Problem 12. Let X be any space with functions. A map $\varphi : \mathbb{P}^n \to X$ is a morphism if and only if $\varphi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X$ is a morphism.

Problem 13. Let $\varphi : X \to Y$ be a morphism of spaces with functions and suppose $Y = \bigcup V_i$ is an open covering such that each restriction $\varphi : \varphi^{-1}(V_i) \to V_i$ is an isomorphism. Then φ is an isomorphism.

Problem 14. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x, y, z]$ of degree 2, then $C \cong \mathbb{P}^1$.

Problem 15. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 16. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y.
(a) Show that X is separated.
(b) Show that $\mathbb{P}^n$ has this property.

Problem 17. [Hartshorne II.2.16 and II.2.17]
Let X be any pre-variety and $f \in k[X]$ a regular function.
(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i.]
(b) $k[D(f)] = k[X]_f$.
(c) Let R be a k-algebra and let $f_1, \dots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \dots, f_r) = R$. If R_{f_i} is a finitely generated k-algebra for each i, then R is a finitely generated k-algebra.
(d) Suppose $f_1, \dots, f_r \in k[X]$ satisfy $(f_1, \dots, f_r) = k[X]$ and $D(f_i)$ is affine for each i. Then X is affine.