GOODIES 2

A morphism of varieties $\varphi : X \to Y$ is *dominant* if $\overline{\varphi(X)} = Y$.

Problem 1. The commutative algebra result *lying over* states that, if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.

(a) Use lying over to show that if $\varphi : X \to Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of Y.

(b) If $\varphi : X \to Y$ is any morphism of varieties, then its image $\varphi(X)$ is *constructible*, i.e. a finite union of locally closed subsets of Y.

Problem 2. Let $m_0, m_1, \ldots, m_N \in k[x_0, \ldots, x_n]$ be all the monomials of degree d. The *Veronese embedding* is the map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by

$$v_d(x_0 : \cdots : x_n) = (m_0(x_0, \ldots, x_n) : \cdots : m_N(x_0, \ldots, x_n)) .$$

(a) Show that v_d is an isomorphism of $\mathbb{P}^n$ with a closed subvariety in $\mathbb{P}^N$.

(b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $S = V_+(f)$ where $f \in k[x_0, \ldots, x_n]$ is an irreducible form of degree d. Show that $S = {v_d}^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

Problem 3. Let L_1, L_2, and L_3 be lines in $\mathbb{P}^3$ such that none of them meet.

(a) There exists a unique quadric surface $S \subset \mathbb{P}^3$ containing L_1, L_2, and L_3. [Hint: Start by applying an automorphism of $\mathbb{P}^3$ to make the lines nice.]

(b) S is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting L_1, L_2, and L_3.

(c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet L_1, L_2, or L_3. Then the number of lines meeting L_1, L_2, L_3, and L_4 is equal to the number of points in $L_4 \cap S$, which is one, two, or infinitely many.

Problem 4. An *algebraic group* is a pre-variety G together with morphisms $m : G \times G \to G$ and $i : G \to G$, and an identity element $e \in G$, such that G is a group in the usual sense when m is used to define multiplication and i maps any element to its inverse element.

(a) Show that $\mathrm{GL}_n(k)$ is an algebraic group.

(b) Show that any algebraic group is separated.

(c) Show that $\mathbb{P}^1$ is not an algebraic group, i.e. it is not possible to find morphisms $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $i : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying the group axioms.

(d) Challenge: How about $\mathbb{P}^n$ for $n \geq 2$?

Problem 5. A morphism $f : X \to Y$ of pre-varieties is called *affine* if, for every open affine subset $V \subset Y$, the inverse image $f^{-1}(V)$ is also affine. The morphism f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for every open affine $V \subset Y$.

(a) Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall\, i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

(b) If f is affine and Y is separated, then X is separated.

Problem 6. Let X be a variety and $V \subset X$ any subset. Then V inherits a structure of space with functions from X. Assume that V is a variety with this structure. Show that V is locally closed in X.

Problem 7. Set $E = V(y^2 - x^3 - 3) \subset \mathbb{C}^2$, $P = (1, 2) \in E$, and $U = E \smallsetminus \{P\}$.
(a) Show that U is an open affine subvariety of E.
(b) Challenge: U is not of the form $D(f)$ for any regular function $f \in \mathcal{O}_E(E)$.

Problem 8. Set $E = k^{n+1}$ and recall that $\mathbb{P}^n = \{\ell \subset E \mid \ell \text{ is a line through the origin of } E\}$. Define $S = \{(\ell, v) \in \mathbb{P}^n \times E \mid v \in \ell\}$, and let $\rho : S \to \mathbb{P}^n$ be the projection.
(a) S is a subbundle of rank 1 of the trivial vector bundle $\mathbb{P}^n \times E$.
Define an $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{L}$ by $\Gamma(U, \mathcal{L}) = \{ \text{ morphisms } s : U \to L \mid \rho s = 1_U\}$.
(b) $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.
Let $\pi : E \smallsetminus \{0\} \to \mathbb{P}^n$ be the projection. For $d \in \mathbb{Z}$ we define an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^n}(d)$ by $\Gamma(U, \mathcal{O}(d)) = \{s \in \mathcal{O}_E(\pi^{-1}(U)) \mid s(\lambda v) = \lambda^d s(v) \; \forall \lambda \in k, v \in E\}$.
(c) The sheaf $\mathcal{O}(d)$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.
(d) Find an integer $d \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(d)$ as an $\mathcal{O}_{\mathbb{P}^n}$-module.