Linear Algebraic Groups Jan. 20, 2026

$k = \bar{k}$ alg. closed field.

Def A LAG is a group G that is is also an affine variety, such that multiplication $\mu : G \times G \to G$ and inverse elt. fcn. $i : G \to G$ are morphisms of varieties.

Challenge: $\frac{1}{2}$ of class do research on alg. geo.

$\frac{1}{2}$ of class does not know what affine variety is!

Example: $G = GL_n$

$G = SL_n = \{ g \in GL_n \mid \det(g) = 1 \}$

$G = O(n) = \{ g \in GL_n \mid g^T g = 1 \}$

Fact: Any subgroup $G \subseteq GL_n$ defined by poly. equs. is a LAG. Every LAG is $\cong$ such a subgroup.

Students w/o alg. geo.:

Ok to ignore discussions about AG. aspects.

LAG = subgp of GL_n def. by poly equs.

Accept: AG $\Rightarrow$ "this map is surjective"

AG $\Rightarrow$ "this vector space has $\dim < \infty$".

Alg. Geo. I, Fall 2026.

Example

$A \in GL_n$ any element.

$k = \bar{k} \Rightarrow A = Q J Q^{-1}$, J Jordan normal form.

$J = D + N$. $D \in GL_n$ diag. $N \in Mat(n \times n)$ nilpotent.

$DN = ND$.

$J = J_s J_u$:

$J_s = D$ <u>semisimple</u> part.

$J_u = D^{-1} J$ <u>unipotent</u> part.

<u>Note</u>: $J_u - I = D^{-1} J - I = D^{-1}(J - D) = D^{-1} N$

$J_u - I$ nilpotent $\Leftrightarrow$ J_u unipotent.

$A = A_s A_u$, $A_s = Q J_s Q^{-1}$ ss part.

$A_u = Q J_u Q^{-1}$ unipot. part.

<u>Fact</u>: A_s, A_u are unique.

If $G \subseteq GL_n$ LAG, then $A_s, A_u \in G$.