LAG 10 2026-02-18

Dual adjoint representation

$\mathrm{Ad}^*: G \longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[G])$

$$\mathrm{Ad}^*(x) = \mathrm{Int}(x^{-1})^* = \lambda(x)\rho(x) : k[G] \longrightarrow k[G].$$

$\mathrm{Ad}^*: G \longrightarrow GL(T_e^*G)$

$$\mathrm{Ad}^*(x) = \mathrm{Int}(x^{-1})^* = \lambda(x)\rho(x) : T_e^*G \longrightarrow T_e^*G$$

$$\mathrm{Ad}^*(x).df(e) = \lambda(x).d(\rho(x).f)(x^{-1})$$

$$= d(\mathrm{Ad}^*(x).f)(e).$$

For $u \in T_e^*G$, $X \in T_eG$: $(\mathrm{Ad}^*(x).u, X) = (u, \mathrm{Ad}(x^{-1}).X)$

because $(\mathrm{Int}(x^{-1})^* u, X) = (u, d\,\mathrm{Int}(x^{-1})_e . X)$

Rationality

$\mu^2 : G \times G \times G \longrightarrow G$ mult.

$(\mu^2)^* f = \sum_i f_i \otimes g_i \otimes h_i$: $f(xyz) = \sum_i f_i(x) g_i(y) h_i(z)$

$(\mathrm{Ad}^*(x).f)(y) = f(\mathrm{Int}(x^{-1})(y)) = f(x^{-1}yx)$

$\mathrm{Ad}^*(x).f = \sum_i f_i(x^{-1}) h_i(x) g_i$

$\mathrm{Ad}^*(x).df(e) = \sum_i f_i(x^{-1}) h_i(x)\, dg_i(e).$

$\therefore$ $\mathrm{Ad}^* : G \longrightarrow GL(T_e^*)$, $\mathrm{Ad} : G \longrightarrow GL(T_eG)$

are rational representations of G.

Lie algebra

G LAG.

$\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G]) = \{\text{tangent vector fields on } G\}$

$D, D' \in \mathcal{D}_G \Rightarrow [D, D'] = DD' - D'D \in \mathcal{D}_G$

$\lambda, \rho : G \longrightarrow \mathrm{Aut}_{k[G]}(\mathcal{D}_G)$

$$\left.\begin{aligned} \lambda(x).D &= \lambda(x)\, D\, \lambda(x^{-1}) \\ \rho(x).D &= \rho(x)\, D\, \rho(x^{-1}) \end{aligned}\right\} \text{ translation of vector fields.}$$

$L(G) = \{ D \in \mathcal{D}_G \mid \lambda(x).D = D \ \forall x \in G \} \subseteq \mathcal{D}_G$ Lie subalg.

<u>Note</u>: $\rho(x).L(G) = L(G)$.

<u>Def</u> $X \in T_eG$, $f \in k[G]$, $y \in G$: $(\overline{X}f)(y) = X(\lambda(y^{-1}).f)$

<u>Lemma</u> $\overline{X} \in L(G)$

<u>Proof</u>

$\overline{X}f \in k[X]$: $\mu^*(f) = \sum g_i \otimes h_i$: $f(xy) = \sum g_i(x) h_i(y)$.

$$\lambda(x^{-1}).f = \sum g_i(x)\, h_i \in k[G]$$

$$X(\lambda(x^{-1}).f) = \sum g_i(x)\, X(h_i) \text{ reg. fcn. of } x \in G.$$

$\overline{X} \in \mathcal{D}_G$: $\overline{X}(fg) = f \cdot (\overline{X}g) + g \cdot (\overline{X}f)$

$\overline{X} \in L(G)$: $(\lambda(x)\, \overline{X}\, \lambda(x^{-1}).f)(y) = (\overline{X}\, \lambda(x^{-1}).f)(x^{-1}y)$

$$= X(\lambda(y^{-1}x)\, \lambda(x^{-1}).f) = X(\lambda(y^{-1}).f) = (\overline{X}f)(y)$$

□

Def $\alpha : D_G \longrightarrow T_eG, \quad (\alpha D).f = (Df)(e)$

Prop $\alpha : L(G) \xrightarrow{\cong} T_eG$ iso. of vector spaces with inverse $X \mapsto \overline{X}$.

Proof

$\alpha(\overline{X}) = X: \quad \alpha(\overline{X}).f = (\overline{X}f)(e) = X(\lambda(e^{-1}).f) = X(f).$

$D \in L(G) \Rightarrow \overline{\alpha D} = D:$

$$(\overline{\alpha D}.f)(x) = (\alpha D)(\lambda(x^{-1}).f) = D(\lambda(x^{-1}).f)(e)$$
$$= (\lambda(x^{-1})D.f)(e) = Df(x).$$

$\square$

Lemma $\alpha \circ \rho(y) \circ \alpha^{-1} = Ad(y) : T_eG \longrightarrow T_eG$

$$\begin{tikzcd}
L(G) \arrow[r, "\alpha"] \arrow[loop below, "\rho(y)"] & T_eG \arrow[loop below, "Ad(y)"]
\end{tikzcd}$$

Proof

$$(\alpha \circ \rho(y) \circ \alpha^{-1})(X)(f) = ((\rho(y).\overline{X}).f)(e)$$
$$= (\rho(y)\overline{X}\rho(y^{-1}).f)(e) = (\overline{X}\rho(y^{-1}).f)(y)$$
$$= X(\lambda(y^{-1})\rho(y^{-1}).f) = (Ad(y).X)(f).$$

$\square$

<u>Lie algebra of subgroup</u>

G LAG, $H \subseteq G$ closed subgroup.

$k[H] = k[G]/I(H)$

$T_eH = \{X \in T_eG \mid X(I(H)) = 0\} \subseteq T_eG$

<u>Def</u>: $\mathcal{D}_{G,H} = \{D \in \mathcal{D}_G \mid D(I(H)) \subseteq I(H)\} \subseteq \mathcal{D}_G$ Lie subalg.

Lie algebra hom: $\phi : \mathcal{D}_{G,H} \longrightarrow \mathcal{D}_H$:

$D \in \mathcal{D}_{G,H}$: $D : k[G] \longrightarrow k[G]$ k-derivation,

$\phi D : k[H] \longrightarrow k[H]$, $(\phi D)(\bar{f}) = \overline{Df}$.

<u>Lemma</u> $\phi : \mathcal{D}_{G,H} \cap L(G) \xrightarrow{\cong} L(H)$ iso. of Lie algebras.

<u>Proof</u>

$$\begin{tikzcd} \mathcal{D}_{G,H} \arrow[r, "\alpha_G"] \arrow[d, "\phi"'] & T_eG \\ \mathcal{D}_H \arrow[r, "\alpha_H"'] & T_eH \arrow[u, "\cup\!\shortmid"'] \end{tikzcd}$$

<u>Note</u>: $x \in H \Rightarrow$ $\lambda(x)(I(H)) \subseteq I(H)$.

$\phi(\mathcal{D}_{G,H} \cap L(G)) \subseteq L(H)$:

$\lambda(x) D = D \lambda(x) : k[G] \longrightarrow k[G] \quad \forall x \in G$

$\Rightarrow \lambda(x)\phi(D) = \phi(D)\lambda(x) : k[G]/I(H) \longrightarrow k[G]/I(H) \quad \forall x \in H$

$$\begin{tikzcd} \mathcal{D}_{G,H} \cap L(G) \arrow[r, "\alpha_G", "\subseteq"'] \arrow[d, "\phi"', "\cap\!\shortmid"] & T_eG \\ L(H) \arrow[r, "\cong", "\alpha_H"'] & T_eH \arrow[u, "\cup\!\shortmid"'] \end{tikzcd}$$

<u>Show</u>: $X \in T_eH \Rightarrow \bar{X} \in \mathcal{D}_{G,H}$

$X \in T_eH$, $f \in I(H)$, $y \in H$:

$(\bar{X}f)(y) = X(\lambda(y^{-1}).f) = 0$ since $\lambda(y^{-1}).f \in I(H)$.

$\therefore$ $\bar{X}(I(H)) \subseteq I(H)$

$\square$

Lie algebra homomorphism

$\phi : G \longrightarrow H$ homomorphism of LAGs.

$d\phi = d\phi_e : L(G) \longrightarrow L(H)$, $d\phi(\overline{X}) = \overline{d\phi_e(X)}$

Lemma

$D \in L(G) \Rightarrow D \circ \phi^* = \phi^* \circ d\phi(D) : k[H] \longrightarrow k[G]$

$$\begin{array}{ccc} k[H] & \xrightarrow{\phi^*} & k[G] \\ {\scriptstyle d\phi(D)}\downarrow & & \downarrow{\scriptstyle D} \\ k[H] & \xrightarrow{\phi^*} & k[G] \end{array}$$

Proof

$X \in T_eG$, $f \in k[H]$, $y \in G$.

$(\overline{X} \circ \phi^*(f))(y) = X(\lambda(y^{-1}).\phi^*(f)) = X(\phi^*(\lambda(\phi(y)^{-1}).f))$

$= d\phi(X)(\lambda(\phi(y)^{-1}).f) = (\overline{d\phi(X)}.f)(\phi(y)) = (\phi^* \circ \overline{d\phi(X)}(f))(y)$

$\square$

Prop $d\phi : L(G) \longrightarrow L(H)$ is a Lie alg. hom.

Proof

$$G \xrightarrow{\phi} \phi(G) \xrightarrow{\subseteq} H$$

$$L(G) \xrightarrow{d\phi} L(\phi(G)) \xrightarrow{\subseteq} L(H)$$

Lie subalg.

WLOG: ϕ surjective $\Rightarrow \phi^* : k[H] \longrightarrow k[G]$ injective.

$\phi^* \circ d\phi([D,D']) = [D,D'] \circ \phi^* = (DD' - D'D) \circ \phi^*$

$= \phi^* \circ (d\phi(D)d\phi(D') - d\phi(D')d\phi(D)) = \phi^* \circ [d\phi(D), d\phi(D')]$

$\Rightarrow d\phi([D,D']) = [d\phi(D), d\phi(D')]$

$\square$