LAG 12 2026-02-26

Thm

$\phi : X \longrightarrow Y$ dominant of irred. varieties.

$r = \dim X - \dim Y$.

$\exists$ dense open $U \subseteq X$ such that:

(1) $\phi \times 1_Z : U \times Z \longrightarrow Y \times Z$ is an open morphism $\forall Z$.

(2) $Y' \subseteq Y$ irred. closed, $X' \subseteq \phi^{-1}(Y')$ irred. comp.,
$X' \cap U \neq \emptyset \Rightarrow \dim(X') = \dim(Y') + r$.

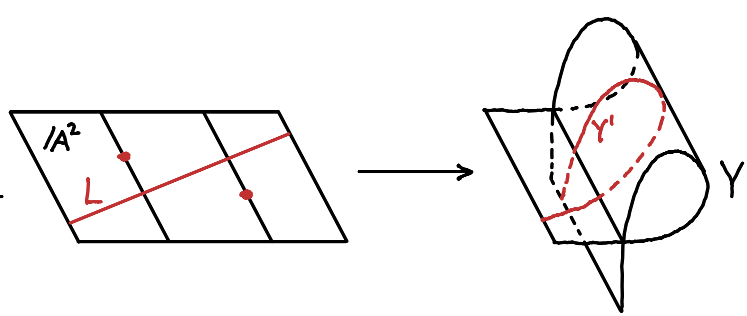
(3) Assume $\dim X = \dim Y$.
$\forall y \in \phi(U) : \# \phi^{-1}(y) = [k(X)_s : k(Y)]$
$k(X)_s = \{ f \in k(X) \mid f \text{ separable } / k(Y) \}$.

Caution: $\phi : X \longrightarrow Y$ dominant, $r = \dim(X) - \dim(Y)$.

True: $y \in Y$ point, $\phi^{-1}(y) \neq \emptyset \Rightarrow \dim \phi^{-1}(y) \geq r$.

False: $Y' \subseteq Y$ closed, irred, $\phi^{-1}(Y') \neq \emptyset \Rightarrow \dim \phi^{-1}(Y') \geq \dim Y' + r$

$X = \mathbb{A}^2 \setminus L$



Integral extensions

A ring, B A-algebra.

$b \in B$ is integral over A if $\exists\ b^n + a_1 b^{n-1} + \dots + a_n = 0$, $a_i \in A$.

B integral over A $\Leftrightarrow$ All elts. integral over A.

B finite over A $\Leftrightarrow$ B f.g. as A-module.

Exer: B finite /A $\Leftrightarrow$ B integral /A & f.g. as A-algebra.

$\bar{A} = \{ b \in B \mid b \text{ integral } /A \} \subseteq B$ subalgebra.

Def A domain A is normal if $A = \bar{A} \subseteq K(A)$.

$\phi : X \longrightarrow Y$ morphism, X, Y affine.

ϕ is finite $\Leftrightarrow$ $k[X]$ is finite over $k[Y]$.

Fact: ϕ finite $\Leftrightarrow$ ϕ is proper with finite fibers
$\Rightarrow$ ϕ is closed with finite fibers.

Y is normal if $k[Y]$ is normal.
non-singular $\Rightarrow$ normal.

Note: Assume X, Y irred. affine, Y normal,
$\phi : X \longrightarrow Y$ finite, birat.
Then $\phi : X \xrightarrow{\cong} Y$ isomorphism.
$k[Y] \subseteq k[X] \subseteq \overline{k[Y]} \subseteq k(Y) = k(X)$.

Zariski's Main Theorem

$\phi : X \longrightarrow Y$ morphism of irred. varieties.
Assume ϕ is bijective and birational, Y normal.
Then ϕ is an isomorphism.

Thm G alg. group. X, Y homogeneous G-varieties.

$\phi: X \longrightarrow Y$ equivariant. $r = \dim(X) - \dim(Y)$.

(a) $\forall Z: \ \phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ is open.

(b) $Y' \subseteq Y$ closed, irred., $X' \subseteq \phi^{-1}(Y')$ irred. comp.

$\Rightarrow \dim(X') = \dim(Y') + r$.

(c) ϕ isomorphism $\Leftrightarrow$ ϕ bijective and $\exists p \in X$:

$d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y$ bijective.

Proof

WLOG G connected, X, Y irred.

(a)+(b) true for $\phi: U \rightarrow Y$, $U \subseteq X$ dense open.

Translate.

(c): $d\phi_p$ surjective $\Rightarrow$ ϕ separable.

ϕ bijective $\Rightarrow$ ϕ birational.

$\phi: U \xrightarrow{\cong} \phi(U)$ for $U \subseteq X$ dense open.

Translate.

□

Cor $\phi: G \longrightarrow G'$ surjective hom. of alg. groups.

(a) $\dim(G) = \dim(G') + \dim \operatorname{Ker}(\phi)$.

(b) ϕ isomorphism $\Leftrightarrow$ ϕ and $d\phi$ are bijective.

Semi-simple automorphisms

G connected LAG. $\mathfrak{g} = L(G) = T_eG$.

$\sigma : G \xrightarrow{\cong} G$ automorphism.

$G_\sigma = \{x \in G \mid \sigma(x) = x\} \subseteq G$ closed subgroup.

$\mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\} \subseteq \mathfrak{g}$ Lie subalg.

Def $\chi : G \longrightarrow G$, $\chi(x) = \sigma(x)x^{-1}$

$G_\sigma = \chi^{-1}(e)$.

$$\chi : G \xrightarrow{(\sigma, i)} G \times G \xrightarrow{\mu} G$$

$$d\chi_e : T_eG \longrightarrow T_eG \oplus T_eG \longrightarrow T_eG$$

$$X \longmapsto (d\sigma(X), -X) \longmapsto d\sigma(X) - X.$$

$L(G_\sigma) \subseteq \mathrm{Ker}(d\chi_e) = \mathfrak{g}_\sigma$

$G \circlearrowright G : g.x = gx$ $\qquad$ $G \circlearrowright G : g \bullet x = \sigma(g)xg^{-1}$.

$\chi : (G, .) \longrightarrow (G, \bullet)$ equivariant morphism.

$\chi(G) = G \bullet e$ is an orbit for $\bullet$ action.

Note: $e \in \overline{\chi(G)}$ non-singular point.

$\chi : G \longrightarrow \overline{\chi(G)}$ separable $\Leftrightarrow$ $d\chi_e(\mathfrak{g}) = T_e\overline{\chi(G)}$

Lemma $L(G_\sigma) = \mathfrak{g}_\sigma \Leftrightarrow d\chi_e(\mathfrak{g}) = T_e\overline{\chi(G)}$

Proof

$$\dim d\chi_e(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{g}_\sigma$$

$$\leq \dim \mathfrak{g} - \dim L(G_\sigma) = \dim G - \dim G_\sigma = \dim \overline{\chi(G)}.$$

$\square$

Def $\sigma : G \xrightarrow{\cong} G$ is <u>semi-simple</u>

$\Leftrightarrow$ $\sigma^* : k[G] \longrightarrow k[G]$ is semi-simple.

Lemma $\sigma : G \xrightarrow{\cong} G$ semi-simple

$\Leftrightarrow$ $\exists\, G \subseteq GL_n$, $s \in GL_n$ semi-simple:

$$\sigma(x) = sxs^{-1} \quad \forall x \in G.$$

Proof $(\Rightarrow)$:

$k[G] = k[f_1, \ldots, f_n]$

$\exists\, \mathrm{Span}_k\{f_1, \ldots, f_n\} \subseteq V' \subseteq k[G]$:

$$\dim(V') < \infty, \quad \sigma^*(V') = V'.$$

$$V = \sum_{x \in G} \rho(x).V' \subseteq k[G]$$

$$\dim(V) < \infty, \quad \rho(x).V = V \quad \forall x \in G.$$

$\sigma^* \rho(x).f = \rho(\sigma^{-1}(x))\, \sigma^*.f \quad \forall f \in k[G]$:

$$(\sigma^* \rho(x).f)(y) = \rho(x).f(\sigma(y)) = f(\sigma(y)x) = f(\sigma(y\sigma^{-1}(x)))$$
$$= (\sigma^* f)(y\sigma^{-1}(x)) = (\rho(\sigma^{-1}(x))\, \sigma^*.f)(y).$$

$\sigma^*.\rho(x).V' = \rho(\sigma^{-1}(x)).V'$

$\therefore\ \sigma^*.V = V.$

$\rho : G \subseteq GL(V)$ closed.

$\sigma^* \rho(x) (\sigma^*)^{-1} = \rho(\sigma^{-1}(x))$

$s = (\sigma^*)^{-1} \in GL(V)$ semi-simple.

$\sigma(x) = sxs^{-1}.$

$\square$