G connected LAG.

__Thm__ Let $\sigma: G \xrightarrow{\cong} G$ be semi-simple.

(1) $\mathcal{X}(G) \subseteq G$ is closed.

(2) $d\mathcal{X}_e : T_e G \longrightarrow\!\!\!\!\!\rightarrow T_e \mathcal{X}(G)$ is surjective.

__Proof__

WLOG $G \subseteq GL(V)$ closed, $s \in GL(V)$ ss.

$\qquad \sigma: GL(V) \longrightarrow GL(V), \quad \sigma(x) = sxs^{-1}.$

$\sigma: End(V) \longrightarrow End(V)$ linear extension.

$d\sigma = \sigma = Ad(s) \in GL(\mathfrak{gl}(V)).$

$G_\sigma = \{x \in G \mid sxs^{-1} = x\}$

$\mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid sXs^{-1} = X\} \subseteq \mathfrak{g} = L(G) \subseteq \mathfrak{gl}(V).$

$\mathcal{X}: GL(V) \longrightarrow GL(V), \quad \mathcal{X}(x) = \sigma(x) x^{-1} = sxs^{-1}x^{-1}.$

__Case $G = GL(V)$:__ $GL(V)_\sigma = \mathfrak{gl}(V)_\sigma \cap GL(V) \Rightarrow T_e(GL(V)_\sigma) = \mathfrak{gl}(V)_\sigma$

$\qquad \Rightarrow d\mathcal{X}_e : T_e GL(V) \longrightarrow\!\!\!\!\!\rightarrow T_e \overline{\mathcal{X}(GL(V))}$ surjective.

Let $X \in T_e \overline{\mathcal{X}(G)} \subseteq T_e \overline{\mathcal{X}(GL(V))}$

$\exists Y \in \mathfrak{gl}(V) : X = d\mathcal{X}_e(Y) = d\sigma(Y) - Y.$

$\sigma(G) = G \Rightarrow d\sigma(\mathfrak{g}) \subseteq \mathfrak{g}$

s semi-simple $\Rightarrow d\sigma \in GL(\mathfrak{gl}(V))$ semi-simple

$\qquad \Rightarrow \exists h \subseteq \mathfrak{gl}(V)$ $d\sigma$-stable, $\mathfrak{gl}(V) = \mathfrak{g} \oplus h.$

$Y = Y' \oplus Y'' \in \mathfrak{g} \oplus h$

$X = d\sigma(Y') - Y' = d\mathcal{X}_e(Y').$

$\therefore d\mathcal{X}_e : T_e G \longrightarrow\!\!\!\!\!\rightarrow T_e \overline{\mathcal{X}(G)}$ surjective.

Show: $\mathcal{X}(G) \subseteq G$ closed.

Def $m(T) = \prod\limits_{\substack{a \text{ eigenval.} \\ \text{of } s^{-1}}} (T-a) \in k[T]$.

$$S = \left\{ y \in GL(V) \;\middle|\; \begin{array}{l} \text{(a)} \; yGy^{-1} = G \\ \text{(b)} \; m(y) = 0 \in End(V) \\ \text{(c)} \; ch.poly(Ad(y)|_{\mathfrak{g}}) = ch.poly(Ad(s^{-1})|_{\mathfrak{g}}) \end{array} \right\}$$

$S \subseteq GL(V)$ closed, $s^{-1} \in S$, all elts. of S are semi-simple.

$y \in S:$ $G_Y = \{x \in G \mid yxy^{-1} = x\}$
$\qquad \mathfrak{g}_Y = \{X \in \mathfrak{g} \mid yXy^{-1} = X\}$

$dim(G_Y) \underset{\uparrow}{=} dim(\mathfrak{g}_Y) \underset{\uparrow}{=} dim(\mathfrak{g}_\sigma) = dim(G_\sigma)$

$\qquad \sigma_Y: G \longrightarrow G \qquad$ (c)
$\qquad x \mapsto yxy^{-1}$

$G \subset S, \quad g \cdot y = gyg^{-1}.$
$\phi_Y: G \longrightarrow G \cdot y, \quad \phi_Y(g) = gyg^{-1}$

$\phi_Y^{-1}(y) = G_Y \implies dim(G \cdot y) = dim(G) - dim(G_Y)$

All orbits have same dimension
$\quad \implies$ all orbits are closed.
$\therefore \mathcal{X}(G) = s(G \cdot s^{-1}) \subseteq G$ is closed.
$\square$

$Z_G(s) = \{x \in G \mid xs = sx\} \subseteq G$ centralizer of $s \in G$.

Cor $s \in G$ semi-simple.

(1) $C = \{xsx^{-1} \mid x \in G\} \subseteq G$ closed.

(2) $G \longrightarrow C$, $x \mapsto xsx^{-1}$ is separable.

(3) $\mathfrak{g} = (Ad(s) - 1)\mathfrak{g} \oplus L(Z_G(s))$

Proof

$\sigma : G \longrightarrow G$, $\sigma(x) = s^{-1}xs$ semi-simple automorphism.

$\chi : G \longrightarrow G$, $\chi(x) = \sigma(x)x^{-1} = s^{-1}xsx^{-1}$.

$\chi(G) \subseteq G$ closed, $\chi : G \longrightarrow \chi(G)$ separable.

$C = s\chi(G)$ closed, $x \mapsto xsx^{-1} = s\chi(x)$ separable.

$G_\sigma = \{s^{-1}xs = x\} = Z_G(s)$.

$L(Z_G(s)) = L(G_\sigma) = \mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\}$

$\qquad = \{X \in T_eG \mid s^{-1}Xs = X\} = \{X \in T_eG \mid sXs^{-1} = X\}$

$\qquad = Ker(Ad(s) - 1) \subseteq \mathfrak{g}.$

s semi-simple $\Rightarrow Ad(s) - 1$ semi-simple

$\qquad \Rightarrow \mathfrak{g} = Im(Ad(s) - 1) \oplus Ker(Ad(s) - 1)$

□

Action by automorphisms

D diagonalizable LAG, G connected LAG.

$D \subset G$ by automorphisms:
- G D-variety.
- $G \xrightarrow{\cong} G$, $g \mapsto d.g$ group hom. $\forall d \in D$.

Differentiate: $T_e G \longrightarrow T_e G$, $X \mapsto d.X$

$\alpha : D \longrightarrow GL(k[G])$ locally rational rep.
$$(\alpha(d).f)(x) = f(d^{-1}x).$$

D diagonalizable $\Rightarrow$ $\alpha(d) : k[G] \longrightarrow k[G]$ semi-simple
$\qquad \Rightarrow g \mapsto d.g$ semi-simple automorphism.

Def $Z_G(D) = \{g \in G \mid d.g = g \ \forall d \in D\} = \bigcap_{d \in D} G_d$

$\qquad Z_{\mathfrak{g}}(D) = \{X \in \mathfrak{g} \mid d.X = X \ \forall d \in D\} = \bigcap_{d \in D} \mathfrak{g}_d$

Note: $L(G_d) = \mathfrak{g}_d$, $L(Z_G(D)) \subseteq Z_{\mathfrak{g}}(D)$.

Cor $L(Z_G(D)) = Z_{\mathfrak{g}}(D)$
Proof
If $D \subset \mathfrak{g}$ trivial: $L(G_d) = \mathfrak{g}_d = \mathfrak{g} \Rightarrow G_d = G$.
$\qquad\qquad\qquad\qquad Z_G(D) = G, \quad Z_{\mathfrak{g}}(D) = \mathfrak{g}$.
Otherwise choose $d \in D$ such that $\mathfrak{g}_d \subsetneq \mathfrak{g}$.
D commutative $\Rightarrow D$ acts on G_d, G_d°.
$Z_G(D) = Z_{G_d}(D) \supseteq Z_{G_d^{\circ}}(D)$, $Z_{\mathfrak{g}}(D) = Z_{\mathfrak{g}_d}(D)$.
Induction on $\dim(G) \Rightarrow$
$\qquad \dim Z_{\mathfrak{g}_d}(D) = \dim Z_{G_d^{\circ}}(D) = \dim Z_{G_d}(D)$

□

$G_s = \{x \in G \mid x \text{ semi-simple}\}$

Commutator: $(x,y) = xyx^{-1}y^{-1}$.

$G \neq e$ nilpotent $\Leftrightarrow$ $Z(G) \neq e$ and $G/Z(G)$ nilpotent

$\Leftrightarrow \exists n \in \mathbb{N} : \forall x_1, \ldots, x_n \in G : (x_1, (x_2, (\cdots (x_{n-1}, x_n) \cdots))) = e$.

Cor G connected nilpotent LAG

$\Rightarrow G_s \subseteq Z(G)$ subgroup.

Proof

$s \in G$ semi-simple.

$\sigma = \text{Int}(s) : G \xrightarrow{\cong} G$

$\chi(x) = \sigma(x) x^{-1} = s x s^{-1} x^{-1} = (s, x)$.

$\chi^n(x) = (s, (s, (\cdots, (s, x) \cdots))) = e$.

$\chi^n(G) = e$.

$d\chi_e = \text{Ad}(s) - 1$.

$(\text{Ad}(s) - 1)^n = (d\chi_e)^n = 0$.

s semi-simple $\Rightarrow \text{Ad}(s) - 1$ ss.

$\therefore \text{Ad}(s) = 1$.

$L(G_\sigma) = \mathfrak{g}_\sigma = \text{Ker}(\text{Ad}(s) - 1) = \mathfrak{g}$

$\Rightarrow G_\sigma = G \Rightarrow \sigma$ trivial $\Rightarrow s \in Z(G)$.

product of commuting ss is ss $\Rightarrow G_s \subseteq Z(G)$ subgroup.

□

<u>Ideal of a closed subgroup</u>

G LAG, $H \subseteq G$ closed subgroup.

$I(H) \subseteq k[G]$ ideal of H.

<u>Lemma</u> $H = \{g \in G \mid p(g).I(H) = I(H)\}$

<u>Proof</u>

$\subseteq$: $g \in H$, $f \in I(H)$, $h \in H \Rightarrow (p(g).f)(h) = f(hg) = 0$

$\supseteq$: $g \in RHS$, $f \in I(H) \Rightarrow f(g) = (p(g).f)(e) = 0$.

□

<u>Lemma</u> $T_e H = \{X \in T_e G \mid \overline{X}.I(H) \subseteq I(H)\}$

<u>Proof</u>

$\mathcal{D}_{G,H} = \{D \in Der_k(k[G], k[G]) \mid D.I(H) \subseteq I(H)\}$

$$
\begin{array}{ccc}
L(G) \cap \mathcal{D}_{G,H} & \xrightarrow{\ \alpha_G\ } & T_e G \\
\cong \Big\downarrow & & \Big\uparrow UI \\
L(H) & \xrightarrow[\cong]{\ \alpha_H\ } & T_e H
\end{array}
$$

Let $X \in T_e G$.

$X \in T_e H \iff \overline{X} \in \mathcal{D}_{G,H}$.

□