LAG 19 2026-03-31

Cor G connected solvable.
$H \subseteq G$ (any) subgroup whose elts. are semi-simple.

(1) $H \subseteq T$ for some max. torus $T \subseteq G$.

(2) $Z_G(H) = N_G(H)$ is connected.

Proof

$H \longrightarrow G/G_u$ injective $\Rightarrow$ H commutative.

If $H \subseteq Z(G)$ then clear (since all max. tori conjugate).

Assume $s \in H$, $s \notin Z(G)$.

$Z_G(s)$ is connected, $H \subseteq Z_G(s) \subsetneq G$.

Induction on $\dim(G)$ $\Rightarrow$

$\exists$ max. torus $T \subseteq Z_G(s)$ with $H \subseteq T$.

$Z_G(H) = Z_{Z_G(s)}(H)$ is connected.

Let $x \in N_G(H)$, $h \in H$.

$xhx^{-1}h^{-1} \in H \cap (G,G) \subseteq H \cap G_u = e$

$\therefore x \in Z_G(H)$.

$\square$

G LAG.

Max. torus in G: subtorus not contained in strictly larger subtorus.

Cor $\Rightarrow$ same def. when G is connected solvable.

__Thm__ G LAG. All max. tori in G are conjugate.

__Proof__

$T, T' \subseteq G$ max. tori.

Choose Borel subgps. $B, B' \subseteq G$ with $T \subseteq B$, $T' \subseteq B'$.

$\exists\, g \in G : \quad gB'g^{-1} = B.$

$T,\ gT'g^{-1} \subseteq B$ max. tori.

$\exists\, b \in B : \quad bgT'g^{-1}b^{-1} = T.$

$\square$

__Lemma__ $T \subseteq G$ max. torus, $H, H' \subseteq G$ conjugate subgps., $T \subseteq H \cap H'$. Then $\exists\, n \in N_G(T) : H' = nHn^{-1}$.

__Proof__

$\exists\, g \in G : H' = gHg^{-1}.$

$g^{-1}Tg,\ T \subseteq H$ max. tori.

$\exists\, h \in H : \ h^{-1}g^{-1}Tgh = T.$

$n = gh \in N_G(T), \quad nHn^{-1} = H'.$

$\square$

Cartan subgroup of LAG G:

$C = Z_G(T)^\circ$ where $T \subseteq G$ max. torus.

Properties

(1) C is nilpotent:

$B \subseteq C$ Borel subgp., $T \subseteq B$.

$B = T \ltimes B_u = T \times B_u$ since $T \subseteq Z(B)$.

B nilpotent $\Rightarrow B = C^\circ = C$.

(2) $T = C_s$ is the only max. torus of C.

(3) $N_G(T) = N_G(C)$.

(4) $N_G(C)/C$ is finite.

Since $N_G(T)/Z_G(T)$ and $Z_G(T)/C$ are finite.

(5) $T \subseteq B \subseteq G$, B Borel $\Rightarrow C \subseteq B$:

$\exists$ Borel subgp. $B' \subseteq G$ with $C \subseteq B'$.

$\exists\, n \in N_G(T)$: $B = nB'n^{-1}$.

$C = nCn^{-1} \subseteq B$.

Lemma G LAG, $S \subseteq G$ subtorus. $\exists\, s \in S$: $Z_G(s) = Z_G(S)$.

Proof

$G \subseteq GL(V)$, $V = \bigoplus V_\chi$, $\chi : S \longrightarrow \mathbb{G}_m$.

Choose $s \in S$ such that

$$0 \neq V_\chi \neq V_{\chi'} \neq 0 \Rightarrow \chi(s) \neq \chi'(s)$$

$Z_G(s) = \{g \in G \mid \forall \chi : g.V_\chi = V_\chi\} = Z_G(S)$.

$\square$

Lemma G LAG, $T \subseteq G$ max. torus, $C = Z_G(T)^\circ$.
$\exists t \in T \ \forall g \in G: \ t \in gCg^{-1} \Rightarrow gCg^{-1} = C$.

Proof
Choose $t \in T$ such that $Z_G(t) = Z_G(T)$.
$t \in gCg^{-1} \Rightarrow g^{-1}tg \in C_s = T$.
$C = Z_G(T)^\circ \subseteq Z_G(g^{-1}tg)^\circ = g^{-1}Cg$.
□

Lemma G LAG, $H \subseteq G$ closed subgp., $X = \bigcup_{g \in G} gHg^{-1} \subseteq G$.
(1) X contains dense open $\subseteq \overline{X}$.
(2) H parabolic $\Rightarrow X = \overline{X} \subseteq G$ is closed.
(3) Assume $N_G(H)/H$ is finite and $\exists h \in H$:
h is in finitely many conjugates of H.
Then $\dim(\overline{X}) = \dim(G)$.

Proof
$A = \{(g,x) \in G \times G \mid x \in gHg^{-1}\} \subseteq G \times G$ closed.
A is H-stable: $(g,x) \in A$, $h \in H \Rightarrow (gh,x) \in A$.
$X = \pi_2(A)$, $\pi_2 : G \times G \longrightarrow G$. (1) + (2) follow from this.
$\dim(A) = \dim(G \times H)$: $G \times H \xrightarrow{\cong} A$, $(g,h) \longmapsto (g, ghg^{-1})$
Assume $N_G(H)/H$ and $\mathcal{H} = \{gHg^{-1} \mid g \in G, h \in gHg^{-1}\}$ finite.
$\mathcal{H} = \{y_1 H y_1^{-1}, \dots, y_n H y_n^{-1}\}$, $y_1, \dots, y_n \in G$.
$$\pi_2^{-1}(h) \cong \{g \in G \mid h \in gHg^{-1}\} = \bigcup_{i=1}^{n} \{g \in G \mid gHg^{-1} = y_i H y_i^{-1}\}$$
$$= \bigcup_{i=1}^{n} y_i N_G(H).$$
$\Rightarrow \dim \pi_2^{-1}(h) = \dim(H)$

$\Rightarrow \dim \pi_2(A) \geq \dim(A) - \dim(H) = \dim(G)$.
□

Thm G connected LAG.

(1) $\forall g \in G \; \exists B \subseteq G$ Borel : $g \in B$.

(2) $\forall s \in G_s \; \exists T \subseteq G$ max. torus : $s \in T$.

(3) The union of all Cartan subgps contains dense open $\subseteq G$.

Proof

$T \subseteq G$ max. torus, $C = Z_G(T)^\circ$.

$N_G(C)/C$ is finite.

$\exists t \in T$: t is in finitely many conjugates of C.

Lemma $\Rightarrow \bigcup_{g \in G} gCg^{-1}$ contains dense open $\subseteq G$.

$\Rightarrow \bigcup_{g \in G} gBg^{-1} = G$ (since larger and closed.)

Let $s \in G$ be semi-simple.

$\exists B \subseteq G$ Borel, $s \in B$.

$\exists T \subseteq B$ max. torus, $s \in T$.

$\square$

Cor G connected LAG, $B \subseteq G$ Borel $\Rightarrow Z(B) = Z(G)$.

Proof

Already proved: $Z(B) \subseteq Z(G)$.

Let $z \in Z(G)$.

$\exists B' \subseteq G$ Borel, $z \in B'$.

$\exists g \in G$: $B = gB'g^{-1}$

$z = gzg^{-1} \in B$.

$\square$