LAG 2 2026-01-22

<u>Algebraic variety</u>: <u>separated</u> SWF with finite open covering by affine varieties.

<u>Irreducible variety</u> X:

$X = X_1 \cup X_2$, $X_i \subseteq X$ closed $\Rightarrow X = X_1$ or $X = X_2$.

$X = \mathrm{Spec}(R)$ affine: X irred $\Leftrightarrow$ R domain.

<u>Connected variety</u> X:

$X = X_1 \cup X_2$, $X_i \subseteq X$ closed, $X_1 \cap X_2 = \emptyset \Rightarrow X = X_1$ or $X = X_2$.

<u>Product of varieties</u> $X \times Y$:

product in category of alg. varieties.

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "\phi_X"] \arrow[drr, bend right, "\phi_Y"'] \arrow[r, dashed, "\exists!"', "\phi"] & X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ & & Y \end{tikzcd}$$

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ as <u>sets</u>!

Example: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ (not product topology!)

$X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ affine

$\Rightarrow X \times Y = \mathrm{Spec}(R \otimes_k S)$.

$f \otimes g \in R \otimes S$: $(f \otimes g)(x,y) = f(x) g(y)$.

<u>Def</u> SWF X is separated

$\Updownarrow$

$\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$ closed.

Projective space

$\mathbb{P}^n = \{ \text{lines through } 0 \text{ in } \mathbb{A}^{n+1} \}$

$\quad = \{ [x_0 : x_1 : \cdots : x_n] \mid (x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\} \}$

Proj. coord. ring: $k[x_0, \ldots, x_n]$.

$f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ homogeneous polys:

$Z(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ closed.

$D_+(x_i) = \{ [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] \} \cong \mathbb{A}^n$.

$\mathbb{P}^n = D_+(x_0) \cup \cdots \cup D_+(x_n)$ alg. var.

Dimension:

X variety.

$$\dim(X) = \max \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists\, X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d \subseteq X \\ \text{s.t. } X_i \text{ closed \& irreducible} \end{array} \right\}$$

$X = \mathrm{Spec}(R)$ irred. affine variety

$\quad \Rightarrow \dim(X) = \mathrm{tr.deg.}_k (K(R))$.

Examples:
- $\dim(\mathbb{A}^n) = \mathrm{tr.deg.}_k\, k(x_1, \ldots, x_n) = n$.
- $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Thm $\phi : X \longrightarrow Y$ morphism of varieties.
Then $\phi(X)$ contains a dense open subset of $\overline{\phi(X)}$.

Fact: $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$.

$$\begin{array}{ccc} \{\text{morphisms } X \longrightarrow Y\} & \longleftrightarrow & \{k\text{-alg. hom. } S \longrightarrow R\} \\ \phi & \longmapsto & \phi^* \end{array}$$

Algebraic Groups

Alg. group: alg. variety G that is also a group:

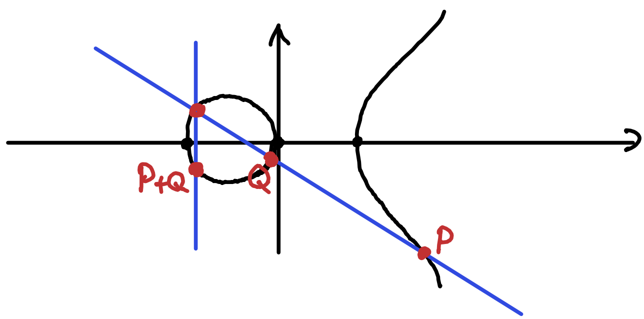
$\mu : G \times G \longrightarrow G$ and $i : G \longrightarrow G$ morphisms.

Consequence: Each elt. $x \in G$ defines two automorphisms:

$$G \xrightarrow{\cong} G, \quad y \longmapsto x \cdot y, \quad y \longmapsto y \cdot x$$

Example: Elliptic curve $E = Z(zy^2 - x^3 + xz^2) \subseteq \mathbb{P}^2$

Draw in $\mathbb{A}^2 = \{z = 1\}$: $\quad y^2 = x^3 - x$



homomorphism of alg. groups $\phi : G \longrightarrow G'$:

morphism + group hom.

products: $G \times G'$

closed subgroup: $H \subseteq G$

LAG: affine algebraic group.

Hopf algebra

G LAG. $\mu : G\times G \longrightarrow G$. $\quad i : G \to G$.

$A = k[G] = A(G) = \mathcal{O}_G(G)$.

$k[G\times G] = A\otimes_k A$.

Comult: $\Delta = \mu^* : A \longrightarrow A\otimes A, \quad f \mapsto f\mu$

$$\Delta f\,(x,y) = f\mu(x,y) = f(x\cdot y)$$

Antipode: $\iota = i^* : A \longrightarrow A, \quad f \mapsto fi$

$$\iota f\,(x) = fi(x) = f(x^{-1})$$

Example

$M_n = \mathrm{Mat}(n\times n, k)$.

$k[M_n] = k[T_{ij},\ 1\le i,j\le n]$.

$G = GL_n = \{x\in M_n \mid \det(x)\neq 0\}$

$A = k[G] = k[M_n]_{\det} = k[T_{ij},\ \det^{-1}]$.

$\mu : G\times G \longrightarrow G$ mult. $\qquad \Delta : A \longrightarrow A\otimes A$

$$(x\cdot y)_{ij} = \sum_k x_{ik}y_{kj}. \qquad \Delta(T_{ij}) = \sum_k T_{ik}\otimes T_{kj}$$

$$\|$$

$$T_{ij}(\mu(x,y)) = \Delta T_{ij}(x,y)$$

$\iota : A \longrightarrow A, \qquad \iota(T_{ij}) = f_{ij} : \quad (x^{-1})_{ij} = f_{ij}(x).$

$$\|$$

$$T_{ij}(i(x)) = \iota T_{ij}(x)$$

Mult: $m : A \otimes A \longrightarrow A, \quad f \otimes g \mapsto fg.$

$m = \delta^*, \qquad \delta : G \longrightarrow G \times G, \quad x \mapsto (x, x).$

$$\delta^*(f \otimes g)(x) = (f \otimes g)(\delta(x)) = f(x)\, g(x) = (fg)(x) = m(f \otimes g)(x).$$

Id. elt: $e \in G. \qquad e : A \longrightarrow k, \quad f \mapsto f(e).$

$\varepsilon : A \xrightarrow{e} k \xrightarrow{\subseteq} A.$

$\varepsilon f(x) = f(e).$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "id \otimes \Delta"] \\
A \otimes A \arrow[r, "\Delta \otimes id"'] & A \otimes A \otimes A
\end{tikzcd}$$

$\Updownarrow$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$$\begin{tikzcd}
A \otimes A \arrow[r, "\iota \otimes id"] & A \otimes A \arrow[d, "m"] \\
A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & A \\
A \otimes A \arrow[r, "id \otimes \iota"'] & A \otimes A \arrow[u, "m"']
\end{tikzcd}$$

$\Updownarrow$

$x^{-1} \cdot x = e = x \cdot x^{-1}.$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "id \otimes e"] \\
A \otimes A \arrow[r, "e \otimes id"'] & A
\end{tikzcd}$$

$\Updownarrow$

$e \cdot x = x = x \cdot e$

Basic results

G alg. group.

Lemma: $\exists!$ irred. comp. $G^\circ \subseteq G$ with $e \in G^\circ$.

G° is closed normal subgroup of finite index.

Pf $X, Y \subseteq G$ irred. comps, $e \in X$, $e \in Y$.

$X \times Y$ irred. $\Rightarrow$ $XY = \mu(X \times Y)$ irred.

$\Rightarrow$ $\overline{XY}$ irred.

$X, Y \subseteq \overline{XY}$ $\Rightarrow$ $X = Y = \overline{XY}$.

X closed under mult.

$i^{-1}(X)$ irred comp., $e \in i^{-1}(X)$ $\Rightarrow$ $i^{-1}(X) = X$.

$\therefore$ $G^\circ = X$ closed subgroup.

$[G : G^\circ]$ = # irred. comps $< \infty$.

$\square$

Cor G irred $\Leftrightarrow$ G connected.

Pf G connected but not irred.

$\Rightarrow \exists x \in G$, x in two irred. comps

$\Rightarrow$ e in two comps.

$$x : G \xrightarrow{\cong} G$$
$$y \mapsto xy$$

$\square$

Cor $H \subseteq G$ closed subgp. with $[G : H] < \infty$

$\Rightarrow$ $G^\circ \subseteq H$.

Pf $H^\circ \subseteq G^\circ$ closed with $[G^\circ : H^\circ] < \infty$.

$G^\circ = x_1 H^\circ \cup x_2 H^\circ \cup \cdots \cup x_\ell H^\circ$. G° irred $\Rightarrow$ $G^\circ = H^\circ$.

$\square$

Lemma $U, V \subseteq G$ dense open subsets $\Rightarrow UV = G$

Proof Let $x \in G$.

$U,\ xV^{-1} \subseteq G$ dense open.

$\Rightarrow U \cap xV^{-1} \neq \emptyset$. $\quad xv^{-1} \in U$ for some $v \in V$.

$\square$

Lemma $H \subseteq G$ any subgroup.

(1) $\overline{H} \subseteq G$ is a closed subgroup. $\quad$ (EXER)

(2) If H contains nonempty open subset of $\overline{H}$, then $H = \overline{H}$. $\quad$ ($H \cdot H = \overline{H}$.)

Prop $\phi : G \longrightarrow G'$ hom. of alg. groups.

(1) $\mathrm{Ker}(\phi) \subseteq G$ closed normal subgp.

(2) $\phi(G) \subseteq G'$ closed subgp

(3) $\phi(G^\circ) = \phi(G)^\circ$

Pf: (2): $\phi(G)$ contains dense open subset of $\overline{\phi(G)}$.

(3): $[\phi(G) : \phi(G^\circ)] < \infty \Rightarrow \phi(G)^\circ \subseteq \phi(G^\circ)$.

$\square$

Prop $\{\phi_i : X_i \longrightarrow G\}_{i\in I}$ family of morphisms.

Assume X_i irred. and $e \in Y_i = \phi_i(X_i)$ $\forall i \in I$.

$H \leq G$ smallest closed subgroup with $Y_i \subseteq H$ $\forall i$

(1) H is connected.

(2) $H = Y_{a(1)}^{\varepsilon(1)} Y_{a(2)}^{\varepsilon(2)} \cdots Y_{a(n)}^{\varepsilon(n)}$ for some $a(1), \ldots, a(n) \in I$, $\varepsilon(1), \ldots, \varepsilon(n) \in \{\pm 1\}$.

Eg. $H = Y_1 Y_2^{-1} Y_3 Y_1 Y_2$

Proof WLOG: $\forall i \in I$ $\exists j \in I$: $Y_i^{-1} = Y_j$.

Given $a = (a(1), \ldots, a(n)) \in I^n$,

set $Y_a = Y_{a(1)} Y_{a(2)} \cdots Y_{a(n)}$.

Then $\overline{Y_a} \leq G$ irred. closed subset.

$Y_b \cdot Y_c = Y_{(b,c)}$.

EXER: $\overline{Y_b} \cdot \overline{Y_c} \subseteq \overline{Y_{(b,c)}}$.

Choose a such that $\dim \overline{Y_a}$ is maximal.

$\forall b$: $\overline{Y_a} \subseteq \overline{Y_a} \cdot \overline{Y_b} \subseteq \overline{Y_{(a,b)}}$.

$\dim \overline{Y_a} = \dim \overline{Y_{(a,b)}}$, both closed irred

$\Rightarrow \overline{Y_a} = \overline{Y_{(a,b)}}$.

$\therefore$ $H = \overline{Y_a} \leq G$ connected closed subgroup.

$\square$