LAG 20 2026-04-02

Thm G connected LAG, $S \subseteq G$ subtorus.

(1) $Z_G(S)$ is connected.

(2) $B \subseteq G$ Borel, $S \subseteq B \Rightarrow Z_G(S) \cap B \subseteq Z_G(S)$ Borel.

All Borel subgps. of $Z_G(S)$ are obtained this way.

Proof

Let $z \in Z = Z_G(S)$. Show: $z \in Z^\circ$.

Choose $B \subseteq G$ Borel, $z \in B$.

$X = \{x.B \in G/B \mid z \in xBx^{-1}\} \subseteq G/B$ closed.

$S \circlearrowright X$, $s.(x.B) = sx.B$ (well def. since $sz = zs$).

Choose $x.B \in X^S \neq \emptyset$.

$\forall s \in S: sx.B = x.B \Leftrightarrow S \subseteq xBx^{-1}$.

Replace $B \mapsto xBx^{-1}$. WLOG $z \in B$ and $S \subseteq B$.

$Z_B(S)$ is connected $\Rightarrow z \in Z_B(S) \subseteq Z^\circ$.

$\therefore Z = Z^\circ$ is connected.

Assume $B \subseteq G$ Borel, $S \subseteq B$.

$Z \cap B = Z_B(S)$ is connected & solvable.

<u>Show</u>: $Z \cap B \subseteq Z$ parabolic.

$Y = \{y \in G \mid y^{-1}Sy \subseteq B\} \subseteq G$ is closed.

$ZB \subseteq Y \Rightarrow \overline{ZB} \subseteq Y$.

$\phi : \overline{ZB} \times S \longrightarrow B/B_u$, $\phi(y,s) = y^{-1}sy.B_u$

Rigidity of diagonalizable groups $\Rightarrow$

$\phi(y,s)$ is independent of y.

$\therefore y^{-1}sy.B_u = s.B_u \ \forall y \in \overline{ZB}, s \in S$.

$\Rightarrow y^{-1}Sy \subseteq SB_u \ \forall y \in \overline{ZB}$.

$B_u \lhd B$ closed connected unipotent normal.

$SB_u \subseteq B$ closed subgp., $S \subseteq SB_u$ max. torus.

Let $y \in \overline{ZB}$

$\exists\, b \in B_u :\ b^{-1}Sb = y^{-1}Sy.$

$yb^{-1} \in N = N_G(S) \Rightarrow y = (yb^{-1})b \in NB.$

$\therefore\ \overline{ZB} \subseteq NB.$

N/Z finite $\Rightarrow N = n_1 Z \cup \cdots \cup n_\ell Z,\ n_1, \dots, n_\ell \in N.$

$\overline{NB} = n_1 \overline{ZB} \cup \cdots \cup n_\ell \overline{ZB} \subseteq NB.$

$\Rightarrow NB \subseteq G$ closed $\Rightarrow NB/B \subseteq G/B$ complete.

$N/N\cap B \longrightarrow NB/B$ bijective equiv. map of hom. N-vars.

$\therefore\ N \cap B \subseteq N$ parabolic.

$(N\cap B)/(Z \cap B)$ finite

$\Rightarrow Z\cap B \subseteq N \cap B$ parabolic

$\Rightarrow Z \cap B \subseteq Z$ parabolic.

$$\begin{array}{ccc} N\cap B & \subseteq & N \\ \cup\!\mid & & \cup\!\mid \\ Z\cap B & \subseteq & Z \end{array}$$

$\therefore\ Z\cap B \subseteq Z$ Borel.

Let $B' \subseteq Z$ be any parabolic subgroup.

$\exists\, z \in Z :\ B' = z(Z\cap B)z^{-1} = Z \cap zBz^{-1}!$

$\square$

<u>Cor</u> G connected LAG, $T \subseteq B \subseteq G$, T max. torus, B Borel.

Then $Z_G(T) = Z_B(T) = N_B(T)$ is connected.

<u>Proof</u>: $Z_G(T) = Z_G(T)^\circ = C \subseteq B.$ $\square$

Thm G connected LAG, $B \subseteq G$ Borel $\Rightarrow N_G(B) = B$.

Proof

Induction on $\dim(G)$.

$H = N_G(B)$. Let $x \in H$. Show: $x \in B$.

$T \subseteq B$ max. torus.

$xTx^{-1} \subseteq B$ also max. torus.

$\exists\, b \in B: \quad bxTx^{-1}b^{-1} = T$.

Replace $x \mapsto bx$: WLOG $xTx^{-1} = T$ and $xBx^{-1} = B$.

$\psi: T \longrightarrow T, \quad t \mapsto xtx^{-1}t^{-1}$ group hom.

Assume $\psi(T) \neq T$:

$S = (\operatorname{Ker}\psi)^\circ \subseteq T$ non-trivial torus.

$x \in Z = Z_G(S)$.

$Z = G$: $\quad \bar{G} = G/S$. $\quad \bar{x} \in N_{\bar{G}}(\bar{B}) = \bar{B} \Rightarrow x \in B$

$Z \neq G$: $\quad x \in N_Z(Z \cap B) = Z \cap B$ since $Z \cap B \subseteq Z$ Borel.

Assume $\psi(T) = T$:

Choose rat. rep. $\phi: G \longrightarrow GL(V)$, $0 \neq v \in V$ such that

$H = \{g \in G \mid \phi(g).v \in kv\}$.

$\exists$ character $\chi: H \longrightarrow \mathbb{G}_m$, $\phi(h).v = \chi(h)v$, for $h \in H$.

$\chi(B_u) = 1, \quad \chi(T) = 1$ since $T \subseteq (H, H)$.

$\Rightarrow \chi(B) = 1$.

$G/B \longrightarrow V, \quad g.B \mapsto \phi(g).v$.

Image is complete, affine, connected.

$\therefore\ \phi(g).v = v \quad \forall g \in G$.

$H = G \quad \Rightarrow \quad B \trianglelefteq G$ normal.

$\Rightarrow G/B$ complete, affine, connected $\Rightarrow B = G$.

$\square$

G connected LAG, $T \subseteq G$ max. torus.

Weyl group: $W = W(G,T) = N_G(T)/Z_G(T)$.

Notation: Given $w \in W$, $\dot{w} \in N_G(T)$ is a representative.

Flag variety: $\mathcal{B} = \{B \subseteq G \text{ Borel}\}$

$G \circlearrowright \mathcal{B}$, $g.B = gBg^{-1}$ (transitive action.)

Isotropy group of $B_0 \in \mathcal{B}$: $G_{B_0} = N_G(B_0) = B_0$.

Identify: $G/B_0 = \mathcal{B}$, $g.B_0 \longleftrightarrow gB_0g^{-1}$.

Note: $\mathcal{B}^T = \{B \in \mathcal{B} \mid T \subseteq B\}$

Cor $W \circlearrowright \mathcal{B}^T$, $w.B = \dot{w}B\dot{w}^{-1}$ simply transitive action.

Proof

$N_G(T) \circlearrowright \mathcal{B}^T$ is transitive.

Isotropy group of B: $B \cap N_G(T) = N_B(T) = Z_G(T)$.

$\square$

Example $G = GL_n$. Borel: $B = \boxed{\begin{smallmatrix} * & * & * \\ 0 & * & * \\ & & * \end{smallmatrix}}$

$F\ell(k^n) = \{(V_1 \subset V_2 \subset \cdots \subset V_n = k^n) \mid dim(V_i) = i\}$

$GL_n \circlearrowright F\ell(k^n)$ transitive.

$E = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset k^n)$ standard flag.

Isotropy group: $G_E = B$.

$GL_n/B \cong F\ell(k^n)$.

Cor G connected LAG, $P \subseteq G$ parabolic.

Then P is connected and $N_G(P) = P$.

Proof

Let $x \in N_G(P)$. Enough to show $x \in P^\circ$.

$B \subseteq P^\circ$ Borel subgroup. $xBx^{-1} \subseteq P^\circ$ also Borel.

$\exists y \in P^\circ : \; xBx^{-1} = yBy^{-1}$.

$y^{-1}x \in N_G(B) = B \Rightarrow x = y(y^{-1}x) \in P^\circ$.

□

Let $T \subseteq B \subseteq P \subseteq G$, T max. torus, B Borel, P parabolic.

Note: $Z_B(T) = Z_P(T) = Z_G(T)$.

$$\Rightarrow \quad W(P,T) = N_P(T)/Z_P(T) \subseteq N_G(T)/Z_G(T) = W(G,T).$$

Flag variety: $\mathcal{P} = \{ gPg^{-1} \mid g \in G \} \cong G/P$.

Cor: $\mathcal{P}^B = \{ P' \in \mathcal{P} \mid B \subseteq P' \} = \{P\}$.

Proof:

Assume $B \subseteq gPg^{-1}$. Then $g^{-1}Bg \subseteq P$.

$\exists p \in P : \; p^{-1}g^{-1}Bgp = B$.

$gp \in N_G(B) = B \Rightarrow gPg^{-1} = (gp)P(gp)^{-1} = P$.

□

Note:
$$\begin{array}{ccccc} G/B & \longrightarrow & G/P, & g.B \mapsto g.P \\ \| & & \| & \\ \mathcal{B} & \longrightarrow & \mathcal{P}, & gBg^{-1} \mapsto gPg^{-1} \end{array}$$

gPg^{-1} = unique conjugate of P containing gBg^{-1}.

Exer: $\mathcal{P}^T = \{ P' \in \mathcal{P} \mid T \subseteq P' \}$.

$W(G,T) \circlearrowright \mathcal{P}^T$, $w.P' = \dot{w}P'\dot{w}^{-1}$ transitive action.

Isotropy group: $W_P = W(P,T)$.

$\therefore \mathcal{P}^T \cong W/W_P$ as W-sets.