LAG 21 2026-04-07

Semi-simple & Reductive

G connected LAG, $T \subseteq G$ max. torus.

$N, N' \trianglelefteq G$ normal $\Rightarrow$ $NN' \trianglelefteq G$ normal.

N, N' also solvable $\Rightarrow$ NN' solvable.

$\therefore \exists!$ max. closed connected solvable normal subgp.

$$R(G) \trianglelefteq G \quad (\text{radical of } G)$$

Unipotent radical: $R_u(G) = R(G)_u$

$R_u(G)$ = unique max. closed connected unipotent normal subgroup of G.

Def G is semi-simple $\Leftrightarrow$ $R(G) = e$.

G is reductive $\Leftrightarrow$ $R_u(G) = e$.

Note: $G/R(G)$ is semi-simple.

$G/R_u(G)$ is reductive.

Note: $R(G) = \left(\bigcap_{B \in \mathcal{B}} B \right)^\circ$

Soon: $R_u(G) = \left(\bigcap_{B \in \mathcal{B}^T} B_u \right)^\circ.$

$\pi : G \longrightarrow \bar{G} = G/R(G)$ semi-simple quotient.

Claim: $\bar{T} = \pi(T) \subseteq \bar{G}$ max. torus.

Proof: $\exists$ max. torus $\bar{S} \subseteq \bar{G}$ with $\bar{T} \subseteq \bar{S}$.

$H \subseteq \pi^{-1}(\bar{S})$ connected closed subgroup.

$H/R(G) \subseteq \bar{S}$ commutative $\Rightarrow$ H solvable.

$H = T \ltimes H_u \Rightarrow \bar{S} = \pi(H) = \pi(T) = \bar{T}$.

$\square$

Weyl groups

$$W = N_G(T)/Z_G(T) \xrightarrow{\pi} N_{\bar G}(\bar T)/Z_{\bar G}(\bar T) = \bar W$$

$$\mathcal{B} = \{B \subseteq G \text{ Borel}\} \underset{\pi}{\xrightarrow{\cong}} \bar{\mathcal{B}} = \{\bar B \subseteq \bar G \text{ Borel}\}$$

$$B \longmapsto \pi(B)$$

$$\pi^{-1}(\bar B) \longleftarrow\!\shortmid\ \bar B$$

$$\begin{tikzcd}
\mathcal{B}^T \arrow[r, "\cong", "\pi"'] & \bar{\mathcal{B}}^{\bar T} \\
\circlearrowleft & \circlearrowleft \\
W \arrow[r, "\pi"] & \bar W
\end{tikzcd}$$

Simply transitive actions

$\Rightarrow \pi : W(G,T) \xrightarrow{\cong} W(\bar G, \bar T)$

<u>Def</u> $\mathrm{rank}(G) = \dim(T)$.

$\mathrm{ssrank}(G) = \mathrm{rank}(G/R(G))$. (semi-simple rank).

- $\mathrm{rank}(G) = 0 \Leftrightarrow G$ unipotent.

<u>Note</u>: $W \subseteq \mathrm{Aut}(T) = \mathrm{Aut}(\mathbb{Z}^n)$, $n = \dim(T)$.

- $\mathrm{ssrank}(G) = 0 \Leftrightarrow G$ solvable $\Rightarrow W = e$.
- $\mathrm{ssrank}(G) = 1 \Rightarrow |W| \leq 2$.

<u>Example</u>: $G = GL_n$.

Borel: $B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$. $B_u = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$ $\quad B/B_u \cong \mathbb{G}_m^n$.

Max torus: $T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$. Opposite Borel: $B^- = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}$

$R(G) \subseteq B \cap B^- = T$. $\quad R_u(G) = e$.

<u>Exer</u>: $R(G) = Z(G) \cong \mathbb{G}_m$.

GL_n is reductive, not semi-simple.

$SL_n \subseteq GL_n$ and $PGL_n = GL_n/Z(GL_n)$ are semi-simple.

<u>Exer</u>: $SL_n = Z(\det -1) \subseteq GL_n$ is connected.

$PGL_n = \mathrm{Aut}(\mathbb{P}^{n-1})$.

Example $G = \begin{bmatrix} * & * \\ & * \end{bmatrix} \subseteq GL_n$. $\quad R_u(G) = \begin{bmatrix} I & * \\ & I \end{bmatrix}$.

$$R(G) = Z(G) \ltimes R_u(G) \cong \mathbb{G}_m \ltimes R_u(G)$$

Notes

(1) X SWF. $X \circlearrowleft G$ action by automorphisms.

$G \circlearrowright \mathcal{O}(X)$, $(g \cdot f)(x) = f(x.g)$.

X/G SWF. $\mathcal{O}(X/G) = \mathcal{O}(X)^G = \{ f \in \mathcal{O}(X) \mid g.f = f \ \forall g \in G \}$.

(2) X affine variety, $X \circlearrowleft \mathbb{G}_m$.

$\mathbb{G}_m \circlearrowright k[X]$ locally rational rep.

$k[X] = \bigoplus_{d \in \mathbb{Z}} k[X]_d$ graded ring.

$k[X]_d = \{ f \in k[X] \mid t.f = t^d f \}$

$\mathcal{O}(X/\mathbb{G}_m) = k[X]_0$

(3) $k[GL_n] = k[x_{ij} \mid 1 \le i,j \le n]_{det}$.

$$k[PGL_n] = k[GL_n]_0$$
$$= k\left[\frac{x_{i_1 j_1} \cdots x_{i_n j_n}}{\det} \;\middle|\; i_\ell, j_\ell \in \{1, 2, \dots, n\},\ 1 \le \ell \le n \right]$$

Example

$\phi : SL_2 \xrightarrow{\subseteq} GL_2 \twoheadrightarrow PGL_2$ group hom.

$\mathrm{Ker}(\phi) = Z(SL_2) = \{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x^2 = 1 \}$.

$\mathrm{Char}(k) \ne 2$: $|Z(SL_2)| = 2$, $PGL_2 = SL_2 / Z(SL_2)$.

$\mathrm{Char}(k) = 2$: $\phi : SL_2 \longrightarrow PGL_2$ bijective, purely inseparable.

Def G LAG, X G-variety.

An <u>equivariant projective embedding</u> of X is an embedding $\psi : X \xrightarrow{\subseteq} \mathbb{P}^n$ together with a alg. group hom. $\rho : G \longrightarrow GL_{n+1}$ such that $\psi(g.x) = \rho(g).\psi(x) \quad \forall x \in X, g \in G$.

Thm (Sumihiro 1973)

G connected LAG, X quasi-projective normal G-variety. Then $\exists$ equiv. proj. embedding of X.

Proof for $X = G/H$, $H \subseteq G$ closed subgroup: LAG 14.

Prop $T \neq e$ torus, X irred. projective T-variety with equiv. proj. embedding.

(1) $\exists\ \mathbb{G}_m \subseteq T : \ X^{\mathbb{G}_m} = X^T$

(2) $|X^T| = 1 \iff X = \{\text{point}\}$.

(3) $|X^T| = 2 \iff \dim(X) = 1$ and $X^T \neq X$.

Cor G connected LAG, $B \subseteq G$ Borel, W Weyl group.

(1) $W = e \iff G$ solvable $\iff \mathrm{ssrank}(G) = 0$.

(2) $|W| = 2 \iff \dim(G/B) = 1 \iff G/B = \mathbb{P}^1 \Leftarrow \mathrm{ssrank}(G) = 1$.

Proof: $W \longleftrightarrow (G/B)^T$.

Proof of Prop:

$X \subseteq \mathbb{P}(V)$ equiv. embedding, $T \longrightarrow GL(V)$ rat. rep.

$V = \bigoplus V_\chi , \quad \chi \in X^*(T) = \{T \longrightarrow \mathbb{G}_m\}$

Choose $\lambda \in X_*(T) = \{\mathbb{G}_m \longrightarrow T\}$:

$$0 \neq V_\chi \neq V_{\chi'} \neq 0 \Rightarrow \langle \lambda, \chi - \chi' \rangle \neq 0.$$

$$\mathbb{P}(V)^T = \coprod \mathbb{P}(V_\chi) = \mathbb{P}(V)^{\lambda(\mathbb{G}_m)} \Rightarrow X^T = X^{\lambda(\mathbb{G}_m)}.$$

WLOG: $T = \mathbb{G}_m$, $V = \bigoplus_{d \in \mathbb{Z}} V_d$, $V_d = \{v \in V \mid t.v = t^d v \ \forall t \in T\}$.

Given $x \in X$, set

$$x_0 = \lim_{t \to 0} t.x \quad , \quad x_\infty = \lim_{t \to \infty} t.x.$$

Note: (a) $x_0, x_\infty \in X^T$, (b) $x_0 = x_\infty \Leftrightarrow x \in X^T$.

$x = [u]$, $u = \sum u_d \in V$. $t.x = [\sum t^d u_d]$

$m = \min\{d : u_d \neq 0\}$, $M = \max\{d : u_d \neq 0\}$.

Then $x_0 = [u_m]$, $x_\infty = [u_M]$.

$\therefore |X^T| = 1 \Leftrightarrow X = \{\text{point}\}$.

Assume $|X^T| = 2$.

Choose $x = [u] \in X \smallsetminus X^T$.

$\ell_m : V \longrightarrow k$ linear, $\ell_m(u_m) \neq 0$, $\ell_m(V_d) = 0$ for $d \neq m$.

$\ell_M : V \longrightarrow k$ linear, $\ell_M(u_M) \neq 0$, $\ell_M(V_d) = 0$ for $d \neq M$.

If $y = [v] \in X \smallsetminus X^T$, then

$y_0 = x_0 \Rightarrow \ell_m(v) \neq 0$ and $y_\infty = x_\infty \Rightarrow \ell_M(v) \neq 0$.

$\phi : X \longrightarrow \mathbb{P}^1$, $\phi([u]) = [\ell_m(u) : \ell_M(u)]$.

T-equiv. morphism: $t.[a:b] = [t^m a : t^M b]$.

$\phi^{-1}(0) \subseteq X$ closed and T-stable.

$(\phi^{-1}(0))^T = \{x_0\} \Rightarrow \phi^{-1}(0) = \{x_0\}$.

$\therefore \dim(X) = 1$.

If $\dim(X) = 1$ and $x \in X \smallsetminus X^T$, then $X = T.x \cup \{x_0, x_\infty\}$.

$\square$