LAG 22 2026-04-09

From now: G connected LAG, $T \subseteq G$ max. torus.

$W = W(G,T) = N_G(T)/Z_G(T)$ Weyl group.

$W \circlearrowright T$: $w.t = \dot{w} t \dot{w}^{-1}$.

$W \circlearrowright X_*(T) = \{\mathbb{G}_m \longrightarrow T\}$: $(w.\lambda)(t) = w.\lambda(t)$.

$W \circlearrowright X^*(T) = \{T \longrightarrow \mathbb{G}_m\}$: $(w.\alpha)(t) = \alpha(w^{-1}.t)$.

Note: $\langle w.\alpha, w.\lambda\rangle = \langle \alpha, \lambda\rangle$.

$\mathfrak{g} = L(G) = T_eG$.

Adjoint action: $G \circlearrowright \mathfrak{g}$, $g.X = Ad(g).X$

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid t.X = \alpha(t)X \ \forall t \in T\}.$$

Note: $S \subseteq T$ subtorus. $Z_G(S) \leq G$ closed conn. subgp.

$$L(Z_G(S)) = \{X \in \mathfrak{g} \mid s.X = X \ \forall s \in S\} = \bigoplus_{\alpha(S)=1} \mathfrak{g}_\alpha.$$

Example: $L(Z_G(T)) = \mathfrak{g}_0$.

Def For $\alpha \in X^*(T)$ define

$T_\alpha = Ker(\alpha)^\circ \subseteq T$ subtorus.

$G_\alpha = Z_G(T_\alpha) \leq G$ closed connected.

$$L(G_\alpha) = \bigoplus_{\beta \in \mathbb{Q}\alpha} \mathfrak{g}_\beta$$

Example $G = GL_n$. $T = \boxed{\begin{smallmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{smallmatrix}}$. $\mathfrak{g} = Mat(n \times n)$

$\alpha = \alpha_{ij} \in X^*(T)$: $\alpha(t) = t_i t_j^{-1}$. $(i \neq j)$.

$\mathfrak{g}_\alpha = kE_{ij}$, $E_{ij} \in \mathfrak{g}$ has (i,j)-entry 1, all others 0.

$T_\alpha = \{t \in T \mid t_i = t_j\}$

$$G_\alpha = \{g \in GL_n \mid g_{pq} \neq 0 \Rightarrow p = q \text{ or } \{p,q\} = \{i,j\}\} \qquad \begin{array}{c|ccccccc|} \hline i \rightarrow & * & & & * & & & \\ & & * & & & & & \\ & & & * & & & & \\ j \rightarrow & * & & & * & & & \\ & & & & & * & & \\ & & & & & & * & \\ & & & & & & & * \\ \hline \end{array}$$

Note: $w \in W$, $\alpha \in X^*(T)$.

$T_{w.\alpha} = w.T_\alpha = \dot{w} T_\alpha \dot{w}^{-1}$

$G_{w.\alpha} = Z_G(\dot{w} T_\alpha \dot{w}^{-1}) = \dot{w} G_\alpha \dot{w}^{-1}$

$\mathfrak{g}_{w.\alpha} = \dot{w}.\mathfrak{g}_\alpha$:

$$X \in \mathfrak{g}_\alpha \Rightarrow t.(\dot{w}.X) = \dot{w}\dot{w}^{-1}t\dot{w}.X = \dot{w}.(\alpha(w^{-1}.t)X)$$
$$= (w.\alpha)(t)\,\dot{w}.X.$$

$P = \{\alpha \in X^*(T) \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$.

Lemma (1) $G = \langle G_\alpha \mid \alpha \in P \rangle$

(2) G solvable $\Leftrightarrow$ G_α solvable $\forall \alpha \in P$.

Proof:

(1) $L(G) = L(\langle G_\alpha \mid \alpha \in P\rangle)$.

(2) $B \subseteq G$ Borel, $T \subseteq B$.

$B \cap G_\alpha \subseteq G_\alpha$ Borel.

G_α solvable $\Leftrightarrow$ $G_\alpha \subseteq B$.

□

Note: $T_\alpha \subseteq Z(G_\alpha)^\circ \subseteq R(G_\alpha) \Rightarrow \operatorname{ssrank}(G_\alpha) \leq 1$.

$\operatorname{ssrank}(G_\alpha) = 1 \Leftrightarrow G_\alpha$ not solvable $\Leftrightarrow |W(G_\alpha, T)| = 2$.

Assume G_α not solvable.

$T \subseteq G_\alpha \subseteq G \Rightarrow W(G_\alpha, T) \subseteq W(G, T)$.

Def (Reflection along α): $s_\alpha \in W$: $W(G_\alpha, T) = \{1, s_\alpha\}$.

Note: $s_\alpha^2 = 1$.

$X_*(T)_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}, \quad X^*(T)_{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}.$

Lemma Assume G_α not solvable.

(1) $\exists! \, \alpha^\vee \in X_*(T)_{\mathbb{R}} : \ s_\alpha . \alpha^\vee = -\alpha^\vee$ and $\langle \alpha, \alpha^\vee \rangle = 2.$

(2) $\lambda \in X_*(T) \Rightarrow s_\alpha . \lambda = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee.$

(3) $\beta \in X^*(T) \Rightarrow s_\alpha . \beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$

Proof

$(1 - s_\alpha) + (1 + s_\alpha) = 2 \in \mathrm{End}_{\mathbb{R}}(X_*(T)_{\mathbb{R}}).$

$(1 - s_\alpha)(1 + s_\alpha) = 1 - s_\alpha^2 = 0.$

$\therefore X_*(T)_{\mathbb{R}} = \mathrm{Ker}(s_\alpha - 1) \oplus \mathrm{Ker}(s_\alpha + 1).$

$\dot{s}_\alpha \in G_\alpha \Rightarrow s_\alpha . t = t \ \forall t \in T_\alpha \Rightarrow X_*(T_\alpha) \subseteq \mathrm{Ker}(s_\alpha - 1).$

$s_\alpha \neq 1 \Rightarrow \dim_{\mathbb{R}} \mathrm{Ker}(s_\alpha + 1) = 1.$

$X_*(T_\alpha) = \alpha^\perp = \{ \lambda \in X_*(T) \mid \langle \alpha, \lambda \rangle = 0 \}.$

$\therefore \exists! \, \alpha^\vee \in \mathrm{Ker}(s_\alpha + 1) : \langle \alpha, \alpha^\vee \rangle = 2.$

$s_\alpha . \lambda = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee :$

true for $\lambda = \alpha^\vee$, true for $\lambda \in X_*(T_\alpha).$

$\beta \in X^*(T), \ \lambda \in X_*(T) \Rightarrow$

$$\langle s_\alpha . \beta, \lambda \rangle = \langle \beta, s_\alpha . \lambda \rangle = \langle \beta, \lambda - \langle \alpha, \lambda \rangle \alpha^\vee \rangle$$

$$= \langle \beta, \lambda \rangle - \langle \alpha, \lambda \rangle \langle \beta, \alpha^\vee \rangle = \langle \beta - \langle \beta, \alpha^\vee \rangle \alpha, \lambda \rangle$$

$\therefore s_\alpha . \beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$

$\square$

W-invariant form

$f : X^*(T)_{\mathbb{R}} \times X^*(T)_{\mathbb{R}} \longrightarrow \mathbb{R}$ any positive definite symmetric bilinear form.

For $\alpha, \beta \in X^*(T)_{\mathbb{R}}$, set $(\alpha, \beta) = \sum_{w \in W} f(w.\alpha, w.\beta)$.

Then $(\cdot,\cdot)$ is W-invariant, but not unique (unless G is semi-simple).

Note: G_α not solvable $\Rightarrow$ $\langle \beta, \alpha^\vee \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

True for $\beta = \alpha$;

$\langle \beta, \alpha^\vee \rangle = 0 \Rightarrow (\beta,\alpha) = (s_\alpha.\beta, s_\alpha.\alpha) = (\beta, -\alpha)$.

$P' = \{\alpha \in P \mid G_\alpha \text{ not solvable}\}$

Prop $W = \langle s_\alpha \mid \alpha \in P' \rangle$

Proof:

Let $1 \neq w \in W$.

Assume $w-1 : X^*(T)_{\mathbb{R}} \xrightarrow{\cong} X^*(T)_{\mathbb{R}}$ is bijective.

Choose $\alpha \in P'$. (Possible since G is not solvable.)

Choose $\beta \in X^*(T)_{\mathbb{R}}$ s.t. $\alpha = (w-1).\beta = w.\beta - \beta$.

$$(\beta,\beta) = (w.\beta, w.\beta) = (\alpha+\beta, \alpha+\beta) = (\alpha,\alpha) + 2(\alpha,\beta) + (\beta,\beta)$$

$$\Rightarrow \langle \beta, \alpha^\vee \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = -1$$

$$\Rightarrow s_\alpha.\beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \alpha + \beta = w.\beta$$

$$\Rightarrow s_\alpha w.\beta = \beta.$$

Replace $w \longmapsto s_\alpha w$.

WLOG: $w-1$ not bijective.

$\psi : T \longrightarrow T$, $\psi(t) = (w^{-1}.t)t^{-1}$ group hom.

<u>Note</u>: $((w-1).\beta)(t) = (w.\beta)(t)\,\beta(t)^{-1} = \beta(\psi(t))$.

$w-1$ not injective $\Rightarrow$ $\psi(T) \subsetneq T$.

$e \neq S = \mathrm{Ker}(\psi)^\circ \subseteq T$ subtorus.

$Z = Z_G(S) \subseteq G$ closed connected.

Induction on $\dim(G)$:

$Z \subseteq G$: $\dot{w} \in N_Z(T) \Rightarrow w \in W(Z,T)$.

Induction $\Rightarrow$ $W(Z,T) = \langle s_\alpha \mid Z_\alpha \text{ not solvable} \rangle$.

<u>Note</u>: $Z_\alpha \subseteq G_\alpha$, $W(Z_\alpha, T) \xrightarrow{\cong} W(G_\alpha, T)$.

$Z = G$: $S \subseteq G$ central torus.

$\pi : G \longrightarrow \bar{G} = G/S$. $\bar{T} = T/S$.

$\pi : W(G,T) \xrightarrow{\cong} W(\bar{G}, \bar{T})$.

Induction $\Rightarrow$ $W(\bar{G}, \bar{T}) = \langle s_{\bar{\alpha}} \mid \bar{G}_{\bar{\alpha}} \text{ not solvable} \rangle$.

Assume $\bar{G}_{\bar{\alpha}}$ not solvable.

<u>Check</u>: $\pi^{-1}(s_{\bar{\alpha}}) \in W(G,T)$ is a reflection.

Choose $n \in N_G(T)$ s.t. $\pi(n) = \dot{s}_{\bar{\alpha}} \in N_{\bar{G}_{\bar{\alpha}}}(\bar{T})$.

$\alpha = \bar{\alpha}\pi : T \longrightarrow \bar{T} \longrightarrow \mathbb{G}_m$. ($G_\alpha$ solvable?)

$t \in T_\alpha \Rightarrow ntn^{-1} = ts$, $s \in S$.

$s_{\bar{\alpha}}^2 = 1 \Rightarrow n^2 \in Z_G(T) \Rightarrow t = n^2 t n^{-2} = ts^2 \Rightarrow s^2 = 1$.

$T_\alpha \longrightarrow \{s \in S \mid s^2 = 1\}$, $t \mapsto ntn^{-1}t^{-1}$

T_α connected, target finite $\Rightarrow$ $n \in Z_G(T_\alpha) = G_\alpha$.

$\therefore$ n defines reflection $s_\alpha \in W(G,T)$.

$\square$