LAG 23 2026-04-14

Lemma U unipotent connected LAG.

$H \subsetneq U$ proper closed conn. subgp. $\Rightarrow$ $H \subsetneq N_U(H)^\circ$.

Proof

$e \neq U$ nilpotent connected $\Rightarrow$ $Z(U)^\circ \neq e$.

$Z(U)^\circ \nsubseteq H$: $H \subsetneq Z(U)^\circ H \leq N_U(H)^\circ$.

$Z(U)^\circ \subseteq H$: $\bar{H} = H/Z(U)^\circ \subsetneq \bar{U} = U/Z(U)^\circ$.

Induction on $\dim(U)$ $\Rightarrow$ $\bar{H} \subsetneq N_{\bar{U}}(\bar{H})^\circ$.

$N_U(H)/Z(U)^\circ = N_{\bar{U}}(\bar{H})$.

$\therefore$ $H \subsetneq N_U(H)^\circ$.

□

Semi-simple groups of rank 1

Assume $\operatorname{rank}(G) = \operatorname{ssrank}(G) = 1$.

$T \subseteq B \subseteq G$, T max. torus, B Borel.

G not solvable, $W = \operatorname{Aut}(T) = \{1, s\}$, $\dim(G/B) = 1$.

$n = \dot{s} \in N_G(T) - T$. $(G/B)^T = \{1.B, n.B\}$.

Note:
- $ntn^{-1} = s.t = t^{-1}$ $\forall t \in T$.
- $n^2 \in Z_G(T)$.

$U^- = nUn^{-1} = n^{-1}Un$.

Lemma Assume $\mathrm{rank}(G) = \mathrm{ssrank}(G) = 1$.

(1) $G - B = UnB$.

(2) $R(G) = (U \cap U^-)^\circ$

(3) $\dim(U/U\cap U^-) = 1$.

Proof

$(G/B)^B = \{1.B\}$.

$Un.B = Bn.B = G/B - \{1.B\}$.

$\therefore\ UnB = G - B$.

Isotropy group: $U_{n.B} = U \cap nBn^{-1} = U \cap U^-$

$$U/U\cap U^- \longrightarrow Un.B \text{ bijective.}$$

$\therefore\ \dim(U/U\cap U^-) = 1$.

$(U\cap U^-)^\circ \subsetneq N_U((U\cap U^-)^\circ)^\circ \Rightarrow (U\cap U^-)^\circ \lhd U$ normal.

$U, T, \{n\} \subseteq N_G((U\cap U^-)^\circ) \Rightarrow (U\cap U^-)^\circ \lhd G$ normal.

$\therefore\ (U\cap U^-)^\circ \subseteq R(G)$.

$\mathrm{rank}(G) = \mathrm{ssrank}(G) \Rightarrow \mathrm{rank}\, R(G) = 0 \Rightarrow R(G)$ unipotent.

$\therefore\ R(G) \subseteq (U\cap U^-)^\circ$.

$\square$

Lemma Assume G semi-simple, $\mathrm{rank}(G) = 1$.

(1) $\dim(U) = 1$, $Z_G(T) = T$, $U \cap U^- = e$.

(2) $L(G) = L(T) \oplus L(U) \oplus L(U^-)$.

(3) $\exists!\ \alpha \in X^*(T)$: $L(U) = \mathfrak{g}_\alpha$, $L(U^-) = \mathfrak{g}_{-\alpha}$.

(4) $U \times B \xrightarrow{\cong} UnB$, $(u, b) \mapsto unb$ iso. of varieties.

Proof

$(U \cap U^-)^\circ = R(G) = e \Rightarrow U \cap U^-$ finite

$\Rightarrow \dim(U) = 1 \Rightarrow \dim(B) = 2 \Rightarrow \dim(G) = 3.$

$T \subseteq Z_G(T) \nsubseteq B$ (since $Z_G(T)$ nilpotent).

$\therefore\ Z_G(T) = T.$

T normalizes $U \cap U^-$ (finite set)

$\Rightarrow T$ centralizes $U \cap U^-$.

$\therefore\ U \cap U^- \subseteq G_u \cap Z_G(T) = e.$

$u : \mathbb{G}_a \longrightarrow U$ iso. $T \circlearrowright U$ by conjugation.

$\exists\, \alpha \in X^*(T) : t\, u(x)\, t^{-1} = u(\alpha(t)x)\ \forall\, t \in T,\ x \in \mathbb{G}_a.$

$\therefore\ L(U) \subseteq \mathfrak{g}_\alpha.$

$U \cap Z_G(T) = e \Rightarrow \alpha \neq 0.$

$T \circlearrowright U^- :\ t\, n u(x) n^{-1} t^{-1} = n\, t^{-1} u(x)\, t\, n^{-1} = n\, u(\alpha(t)^{-1}x)\, n^{-1}$

$\therefore\ L(U^-) \subseteq \mathfrak{g}_{-\alpha}.$

$\dim(G) = 3 \Rightarrow L(G) = L(T) \oplus L(U) \oplus L(U^-).$

$U^- \times B \circlearrowright G,\ (u,b).g = ugb^{-1}.$

$\phi : U^- \times B \longrightarrow G,\ \phi(u,b) = ub^{-1}$ equivariant map.

$\phi^{-1}(e) = U^- \cap B = U^- \cap U = e \Rightarrow \phi$ injective.

$d\phi : L(U^-) \oplus L(B) \xrightarrow{\cong} L(G),\ (X,Y) \longmapsto X - Y.$

$\therefore\ \phi : U^- \times B \xrightarrow{\cong} U^- B \subseteq G$ iso. of vars.

$\Rightarrow U \times B \xrightarrow{\cong} UnB$ (since $U^- = n^{-1}Un$).

$\square$

<u>Thm</u> Let G be semi-simple of rank 1.

$\exists\, u : \mathbb{G}_a \xrightarrow{\cong} B_u$, $\alpha^\vee : \mathbb{G}_m \longrightarrow T$, $\alpha : T \longrightarrow \mathbb{G}_m$,
$n \in N_G(T) - T$ such that

(1) $\langle \alpha, \alpha^\vee \rangle = 2$

(2) $n^2 = \alpha^\vee(-1)$

(3) $t\,u(x)\,t^{-1} = u(\alpha(t)\,x) \quad \forall\, t \in T,\ x \in \mathbb{G}_a$

(4) $n\,u(y)\,n^{-1} = u(-y^{-1})\,n\,\alpha^\vee(y)\,u(-y^{-1}) \quad \forall\, y \in \mathbb{G}_m = \mathbb{G}_a - \{0\} = K^\times$.

<u>Example</u>: $G = SL_2$ and $G = PGL_2$:

$u(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $\alpha^\vee(s) = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}$, $\alpha\left(\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}\right) = st^{-1}$, $n = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$G = SL_2 \Leftrightarrow \alpha^\vee$ isomorphism. $G = PGL_2 \Leftrightarrow \alpha$ isomorphism.

<u>Note</u>: Given $T \subset B \subset G$:

α and $\alpha^\vee$ are unique: $L(B_u) = \mathfrak{g}_\alpha$.

$u : \mathbb{G}_a \xrightarrow{\cong} B_u$ can be chosen arbitrarily.

n depends on u.

<u>Note</u>: Given $t \in T$, can also use

$u'(y) = t\,u(y)\,t^{-1} = u(\alpha(t)u)$ and $n' = t^2 n$.

Application

Idea: $B = \{s\,u(x)\} \cong \mathbb{G}_m \times \mathbb{G}_a$ (as variety)

$$G - B = B u n B = \{u(y)\,n\,t\,u(z)\} \cong \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a.$$

Def Given $m \in \{1,2\}$, define

$H^{(m)} = (\mathbb{G}_m \times \mathbb{G}_a) \amalg (\mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a)$

with binary operation:

$(s,x)\cdot(t,z) = (st, t^{-m}x + z)$

$(s,x)\cdot(y,t,z) = (s^m(x+y), s^{-1}t, z)$

$(0,1,0)\cdot(t,z) = (0,t,z)$

$$(0,1,0)\cdot(y,t,z) = \begin{cases} ((-1)^{2/m}t, z) & \text{if } y = 0 \\ (-y^{-1}, (-y)^{2/m}t, -y^{-1}t^{-m}+z) & \text{if } y \neq 0 \end{cases}$$

$(v,s,x)\cdot h = (1,v)\cdot(0,1,0)\cdot(s,x)\cdot h, \quad h \in H^{(m)}.$

Let $G, u, \alpha^\vee, \alpha, n$ be as in Thm. Define maps of sets:

$\phi : H^{(2)} \longrightarrow G$; $\quad (s,x) \mapsto \alpha^\vee(s)\,u(x)$

$\qquad (v,s,x) \mapsto u(v)\,n\,\alpha^\vee(s)\,u(x)$

$\psi : G \longrightarrow H^{(1)}$; $\quad t\,u(z) \mapsto (\alpha(t), z)$

$\qquad u(y)\,n\,t\,u(z) \mapsto (y, \alpha(t), z)$

Exer: (1) - (4) $\Rightarrow$ $\phi(h_1 \cdot h_2) = \phi(h_1)\phi(h_2)$

$\psi(g_1 g_2) = \psi(g_1)\cdot\psi(g_2)$

$\underline{\text{Note}}$: $\mathbb{G}_m \xrightarrow{\alpha^\vee} T \xrightarrow{\alpha} \mathbb{G}_m$

$\langle \alpha, \alpha^\vee \rangle = 2 \Rightarrow \alpha^\vee$ iso. or α iso.

<u>Assume $\alpha^\vee$ isomorphism:</u>

$\phi : \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a \xrightarrow{\cong} G - B$ iso. of varieties.

$\phi : H^{(2)} \longrightarrow G$ bijective.

$\therefore SL_2 \xrightarrow{\phi^{-1}} H^{(2)} \xrightarrow{\phi} G$ iso. of LAGs.

<u>Assume α isomorphism:</u>

$\psi : G - B \xrightarrow{\cong} \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_a$ iso. of varieties.

$\psi : G \longrightarrow H^{(1)}$ bijective.

$\therefore G \xrightarrow{\psi} H^{(1)} \xrightarrow{\psi^{-1}} PGL_2$ iso. of LAGs.