LAG 24 2026-04-16

Proof of Thm

Identify $T = \mathbb{G}_m = \mathbb{G}_a - \{0\} = k^\times$.

Choose $u : \mathbb{G}_a \xrightarrow{\cong} B_u$.

For some $m \in \mathbb{Z}$: $t\,u(x)\,t^{-1} = u(t^m x)$ $\forall t \in T,\ x \in \mathbb{G}_a$.

<u>WLOG</u>: $m > 0$. (Replace $t \mapsto t^{-1}$)

Choose $n \in N_G(T) - T$.

$\bar{n} \in W(G,T)$ has order 2 $\Rightarrow$

$$n^2 \in T \quad \text{and} \quad n t n^{-1} = t^{-1} \quad \forall t \in T.$$

$\therefore\ n^2 = n\, n^2 n^{-1} = n^{-2} \Rightarrow n^4 = 1.$

$$\mathbb{G}_a \times T \times \mathbb{G}_a \xrightarrow{\cong} B_u n B = G - B$$
$$(x, t, z) \longmapsto u(x)\, n\, t\, u(z)$$

$B_u \cap n B_u n^{-1} = e \Rightarrow n\, u(y)\, n^{-1} \in G - B$ $\forall y \in T$.

$\exists\, (f, g, h) : T \longrightarrow \mathbb{G}_a \times T \times \mathbb{G}_a$

$$n\, u(y)\, n^{-1} = u(f(y))\, n\, g(y)\, u(h(y)) \quad \forall y \in T.$$

<u>Note</u>: $f(1) \neq 0$, since otherwise

$$u(1)\, n^{-1} = g(1)\, u(h(1)) \Rightarrow n \in B \ \lightning$$

$$\begin{aligned} t\, n\, u(y)\, n^{-1} t^{-1} &= t\, u(f(y))\, n\, g(y)\, u(h(y))\, t^{-1} \\ &= u(t^m f(y))\, t n\, t^{-1} g(y)\, u(t^m h(y)) \end{aligned}$$

Choose $t \in \mathbb{G}_m$ such that $t^m f(1) = -1$.

<u>Replace</u>:

$$\begin{aligned} n &\longmapsto tn \\ f &\longmapsto [y \mapsto t^{m} f(y)] \\ g &\longmapsto [y \mapsto t^{-1} g(y)] \\ h &\longmapsto [y \mapsto t^{m} h(y)] \end{aligned}$$

<u>WLOG</u>: $f(1) = -1$ and

$$n\, u(y)\, n^{-1} = u(f(y))\, n\, g(y)\, u(h(y)).$$

$$\begin{aligned} & u(f(-y))\, n\, g(-y)\, u(h(-y)) \\ &= n\, u(-y)\, n^{-1} = (n\, u(y)\, n^{-1})^{-1} \\ &= (u(f(y))\, n\, g(y)\, u(h(y)))^{-1} \\ &= u(-h(y))\, g(y)^{-1}\, n^{-1}\, u(-f(y)) \\ &= u(-h(y))\, n\, n^{2} g(y)\, u(-f(y)) \end{aligned}$$

$\therefore\ f(-y) = -h(y)$ and $g(-y) = n^{2} g(y)$.

$$\begin{aligned} & u(f(t^{m} y))\, n\, g(t^{m} y)\, u(h(t^{m} y)) \\ &= n\, u(t^{m} y)\, n^{-1} = t^{-1}\, n\, u(y)\, n^{-1}\, t \\ &= t^{-1}\, u(f(y))\, n\, g(y)\, u(h(y))\, t \\ &= u(t^{-m} f(y))\, n\, t^{2} g(y)\, u(t^{-m} h(y)) \end{aligned}$$

For $s = t^{m}$, $y = 1$: $f(s) = s^{-1} f(1) = -s^{-1}$.

$\qquad\qquad\qquad\qquad\quad h(s) = -f(-s) = -s^{-1}$.

$$\therefore\ n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, g(y)\, u(-y^{-1})$$

$$u(-(y+1)^{-1})\, n B = n\, u(y+1)\, n^{-1} B$$
$$= n\, u(y) n^{-1} n\, u(1) n^{-1} B$$
$$= u(-y^{-1})\, n\, g(y)\, u(-y^{-1})\, u(-1)\, n B$$
$$= u(-y^{-1})\, g(y)^{-1} n\, u(-y^{-1}-1)\, n^{-1} B$$
$$= u(-y^{-1})\, g(y)^{-1} u((y^{-1}+1)^{-1})\, n B$$
$$= u(-y^{-1})\, u(g(y)^{-m}(y^{-1}+1)^{-1})\, n B$$

$$\therefore\ -(y+1)^{-1} = -y^{-1} + g(y)^{-m}(y^{-1}+1)^{-1}$$
$$\Rightarrow\ g(y)^{-m} = (y^{-1}-(y+1)^{-1})(y^{-1}+1) = y^{-2}$$

$m = 1$: $\quad g(y) = y^2$

$m = 2$: $\quad g(y)^2 = y^2, \quad g(y) = \varepsilon y, \quad \varepsilon = \pm 1.$

$$n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, \varepsilon y\, u(-y^{-1})$$

Note: $\varepsilon\, u(y)\, \varepsilon^{-1} = u(\varepsilon^2 y) = u(y).$

Replace: $n \mapsto n\varepsilon$. WLOG: $g(y) = y$.

<u>Define</u>: $\alpha : T \longrightarrow \mathbb{G}_m, \quad \alpha(t) = t^m$

$\alpha^\vee : \mathbb{G}_m \longrightarrow T, \quad \alpha^\vee(t) = g(t) = t^{2/m}$

<u>Now</u>:
- $\langle \alpha, \alpha^\vee \rangle = 2$
- $t\, u(x)\, t^{-1} = u(\alpha(t)\, x)$
- $n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1})$

$$(-y)^{2/m} = g(-y) = n^2 g(y) = n^2 y^{2/m}.$$

$m = 1$: $\quad n^2 = 1 = \alpha^\vee(-1). \qquad m = 2$: $\quad n^2 = -1 = \alpha^\vee(-1).$

$\square$

Prop G reductive LAG.

(1) $R(G) = Z(G)^\circ$ is a central torus.

(2) $R(G) \cap (G,G)$ is finite.

Proof

$R(G)$ connected solvable, $R(G)_u = e \Rightarrow R(G)$ torus.

$Z_G(R(G))^\circ = N_G(R(G))^\circ = G \Rightarrow R(G) \subseteq Z(G)$.

$Z(G)^\circ \subseteq G$ closed conn. solvable normal $\Rightarrow Z(G)^\circ \subseteq R(G)$.

$G \subseteq GL(V)$ closed.

$$V = \bigoplus_{\chi} V_\chi, \quad \chi \in X^*(R(G)).$$

$v \in V_\chi,\ g \in G,\ s \in R(G) \Rightarrow s.(g.v) = g.(s.v) = \chi(s)\, g.v$

$\therefore\ G.V_\chi = V_\chi$.

Let $g \in R(G) \cap (G,G)$.

$g : V_\chi \longrightarrow V_\chi$ mult. by $\chi(g)$.

$g \in (G,G) \Rightarrow 1 = \det(g : V_\chi \to V_\chi) = \chi(g)^{\dim(V_\chi)}$.

$\square$

Note: G not solvable $\Rightarrow \dim(G) \geq 3$.

Must have $e \neq T \subsetneq B \subsetneq G$.

Reductive of semi-simple rank 1

Assume G is reductive, $\mathrm{ssrank}(G) = 1$.

Fix $T \subseteq B \subseteq G$, T max. torus, B Borel.

(1) $\dim(G/T) = 2$, $\dim(G/B) = 1$.

$G/R(G)$ semi-simple of rank 1.

$\dim(G/R(G)) = 3$. $\dim(T/R(G)) = 1$.

(2) $Z_G(T) = T$.

$T \subseteq Z_G(T) \subsetneq B$.

(3) (G,G) semi-simple of rank 1.

(G,G) not solvable $\Rightarrow \dim(G,G) \geq 3$.

$(G,G) \longrightarrow G/R(G)$ finite and surjective.

$R((G,G)) \longrightarrow \{e\} \Rightarrow R((G,G)) = e$.

Must have $\mathrm{rank}(G,G) = 1$.

(4) $T_1 = (G,G) \cap T$, $B_1 = (G,G) \cap B = T_1 \ltimes B_u$.

$T_1 \subseteq B_1 \subseteq (G,G)$ max. torus, Borel.

$R(G) \subseteq T$. $(G,G) \twoheadrightarrow G/T \twoheadrightarrow G/B$.

$\dim((G,G) \cap T) = 1$, $\dim((G,G) \cap B) = 2$.

$T_1^\circ \subseteq B_1^\circ \subseteq (G,G)$ max. torus, Borel. ($B_1 := (G,G) \cap B$.)

$T_1 \subseteq Z_{(G,G)}(T_1^\circ) = T_1^\circ$.

$\dim B_u = \dim(B_1^\circ)_u = 1 \Rightarrow B_u = (B_1^\circ)_u \subseteq (G,G)$.

$B_1 = (G,G) \cap (T \ltimes B_u) = T_1 \ltimes B_u$.

<u>Note</u>: $T = T_1 R(G)$, $B = B_1 R(G)$.

Choose $u : \mathbb{G}_a \xrightarrow{\cong} B_u$.

$\exists!\, \alpha \in X^*(T)$: $t\,u(x)\,t^{-1} = u(\alpha(t)x) \quad \forall\, t \in T,\ x \in \mathbb{G}_a$.

(5) $L(G) = L(T) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$; $\quad \mathfrak{g}_\alpha = L(B_u)$.

$B = T \ltimes B_u$. $\quad L(B) = L(T) \oplus L(B_u)$.

<u>Def</u>: $R(G,T) = \{\alpha, -\alpha\}$ <u>roots</u> of (G,T).

<u>Note</u>: $\alpha \in X^*(T)$ root of (G,T) $\Leftrightarrow$ $\alpha \neq 0$ and $L(G)_\alpha \neq 0$.

(6) $L(G,G) = L(T_1) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

(7) $W((G,G), T_1) \xrightarrow{\cong} W(G,T)$

$s_\alpha \in W$ represented by $\dot{s}_\alpha \in N_{(G,G)}(T_1)$.

(8) $\alpha^\vee \in X_*(T_1) \subseteq X_*(T)$:

Def. $\alpha^\triangledown : \mathbb{G}_m \longrightarrow T_1$ by $\langle \alpha, \alpha^\triangledown \rangle = 2$.

Show: $s_\alpha . \alpha^\triangledown = -\alpha^\triangledown$.

$(s_\alpha . \alpha^\triangledown)(t) = s_\alpha . (\alpha^\triangledown(t)) = \dot{s}_\alpha \alpha^\triangledown(t) \dot{s}_\alpha^{-1} = \alpha^\triangledown(t)^{-1}$.

(9) $\exists\, n \in N_{(G,G)}(T_1) - T_1$ such that:

$n^2 = \alpha^\vee(-1)$ and

$n\, u(y)\, n^{-1} = u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1}) \quad \forall\, y \in \mathbb{G}_m$.