LAG 25 2026-04-23

<u>Note</u>: Assume $T \subseteq B \subseteq G$, B Borel.

Any character $\chi \in X^*(T)$ extends uniquely to $\chi : B \longrightarrow \mathbb{G}_m$.

$B = T \ltimes B_u$. $\chi(tu) = \chi(t)$.

<u>Prop</u> G semi-simple of rank 1, $T \subseteq B \subseteq G$, B Borel,

$L(B) = L(T) \oplus L(G)_\alpha$.

Assume $\chi \in X^*(T)$, $0 \neq f \in k[G]$ satisfy:

$\forall g \in G,\ b \in B$: $f(gb) = \chi(b) f(g)$. Then:

(1) $\langle \chi, \alpha^\vee \rangle \geq 0$.

(2) $\langle \chi, \alpha^\vee \rangle = 0 \Leftrightarrow f$ constant.

<u>Proof</u>

$n B_u n^{-1} B \subseteq G$ dense subset $\Rightarrow$ f not zero on $n B_u n^{-1}$.

$$\underbrace{f(n\, u(y)\, n^{-1})}_{\mathrm{poly}(y) \neq 0} = f(u(-y^{-1})\, n\, \alpha^\vee(y)\, u(-y^{-1}))$$

$$= y^{\langle \chi, \alpha^\vee \rangle} \underbrace{f(u(-y^{-1})\, n)}_{\mathrm{poly}(y^{-1})}.$$

$\therefore$ $\langle \chi, \alpha^\vee \rangle \geq 0$.

$\langle \chi, \alpha^\vee \rangle = 0 \Rightarrow f(n\, u(y)\, n^{-1})$ constant

$\Rightarrow f$ constant $\Rightarrow \langle \chi, \alpha^\vee \rangle = 0$.

$\square$

Assume $\mathrm{ssrank}(G) = 1$:

$T \subset B \subset G$ max. torus, Borel.

$$\begin{tikzcd}
G \arrow[r, "\pi"] & \overline{G} = G/R_u(G) & \\
B \arrow[u, phantom, "\cup"] \arrow[r] & \overline{B} = B/R_u(G) \arrow[u, phantom, "\cup"] & \\
T \arrow[u, phantom, "\cup"] \arrow[r, "\cong", "\pi"'] & \overline{T} \arrow[u, phantom, "\cup"] \arrow[r, "\overline{\alpha}"] & \mathbb{G}_m \\
& \mathbb{G}_m \arrow[u, "\overline{\alpha}^\vee"'] &
\end{tikzcd}$$

$R(\overline{G}, \overline{T}) = \{\pm \overline{\alpha}\}$.

<u>Def</u>: $R(G,T) = \{\pm \alpha\}$, $\alpha = \overline{\alpha}\,\pi$.

<u>Note</u>: Given $\beta \in X^*(T)$:

$\beta \in R(G,T) \Leftrightarrow \beta \neq 0$ and $L(G/R_u(G))_\beta \neq 0$.

<u>Note</u>: $\alpha^\vee = \pi^{-1}\overline{\alpha}^\vee \in X_*(T)$.

Roots

G connected LAG. $T \subseteq G$ max. torus.

Given $\alpha \in X^*(T)$:

$T_\alpha = \mathrm{Ker}(\alpha)^\circ \subseteq T$, $G_\alpha = Z_G(T_\alpha)$, $R_\alpha = R_u(G_\alpha)$.

<u>Recall</u>: $\mathrm{ssrank}(G_\alpha) \leq 1$.

<u>Def</u>: $R(G,T) = \{\alpha \in X^*(T) \mid \alpha \neq 0 \text{ and } L(G_\alpha/R_\alpha)_\alpha \neq 0\}$.

Lemma $\alpha \neq \beta \in R(G,T) \Rightarrow \alpha^\vee \neq \beta^\vee$.

Proof

Assume $\alpha^\vee = \beta^\vee$.

Let $(-,-)$ be W-invariant, sym., pos. definite.

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \langle \alpha, \beta^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2 = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$$

$$\Rightarrow \quad (\alpha,\alpha) = (\alpha,\beta) = (\beta,\beta).$$

$$(\alpha-\beta, \alpha-\beta) = (\alpha,\alpha) - 2(\alpha,\beta) + (\beta,\beta) = 0.$$

$\square$

Root datum: $\Psi = \Psi(G,T) = (X, R, X^\vee, R^\vee)$

$X = X^*(T) \cong \mathbb{Z}^n$ lattice.

$R = R(G,T) \subseteq X$.

$X^\vee = X_*(T)$ dual lattice.

$R^\vee = \{\alpha^\vee \mid \alpha \in R\}$

Implicit: Bijection $R \longleftrightarrow R^\vee$, $\alpha \leftrightarrow \alpha^\vee$.

Properties/Axioms

(1) $\langle \alpha, \alpha^\vee \rangle = 2 \quad \forall \alpha \in R$.

(2) $s_\alpha . R = R$ and $s_\alpha . R^\vee = R^\vee$.

True since $G_{s_\alpha . \beta} = \dot{s}_\alpha G_\beta \dot{s}_\alpha^{-1}$.

$\Psi(R,T)$ is **reduced**: $\alpha \in R \Rightarrow \mathbb{R}\alpha \cap R = \{\pm \alpha\}$.

True since $\beta \in \mathbb{R}\alpha \Rightarrow T_\beta = T_\alpha \Rightarrow G_\beta = G_\alpha$.

Notes

- R is a root system in $\mathrm{Span}_{\mathbb{R}}(R) \subseteq X^*(T)_{\mathbb{R}}$.
- Dual root datum $(X^\vee, R^\vee, X, R)$ satisfies same axioms.
- Weyl group of $(X, R, X^\vee, R^\vee)$: $W = \langle s_\alpha : \alpha \in R \rangle \subseteq \mathrm{Aut}(X)$.

Positive roots

Let $\alpha \in R$. $L(G_\alpha/R_\alpha) = L(T) \oplus k_\alpha \oplus k_{-\alpha}$

$\mathcal{B}(G_\alpha)^T = \{B_\alpha, B_{-\alpha}\}$.

$B_\alpha \subseteq G_\alpha$ unique Borel s.t. $T \subseteq B_\alpha$ and $L(B_\alpha/R_\alpha)_\alpha \neq 0$.

Let $B \subseteq G$ Borel, $T \subseteq B \subseteq G$.

$R^+(B) = \{\alpha \in R \mid B_\alpha \subseteq B\}$, $R^-(B) = \{\alpha \in R \mid B_{-\alpha} \subseteq B\}$.

Note:
- $R(G,T) = R^+(B) \amalg R^-(B)$
- $\alpha \in R^+(B) \Leftrightarrow B_\alpha = G_\alpha \cap B \Leftrightarrow L(G_\alpha \cap B/R_\alpha)_\alpha \neq 0$.

Prop $R^+(B) \subseteq R(G,T)$ is a system of positive roots:

$$\exists\, \chi \in X^*(T) : R^+(B) = \{\alpha \in R \mid \langle \chi, \alpha^\vee \rangle > 0\}.$$

Proof

$G/B \subseteq \mathbb{P}(V)$ equiv. proj. embedding.

$G \longrightarrow GL(V)$ rat. rep., $1.B = [v] \in \mathbb{P}(V)^B$.

$\exists\, \chi : B \longrightarrow \mathbb{G}_m$: $b.v = \chi(b)v \ \forall b \in B$.

Choose $\ell : V \longrightarrow k$ linear such that

$\ell(n_\alpha . v) \neq \ell(v) \ \forall \alpha \in R$ where $n_\alpha \in (G_\alpha, G_\alpha) \setminus B$.

Def. $F : G \longrightarrow k$, $F(g) = \ell(g.v)$.

$F \in k[G]$ and $F(gb) = \chi(b) F(g) \ \forall g \in G, b \in B$.

Let $\alpha \in R^+(B)$. $\overline{G}_\alpha = G_\alpha/R_\alpha$ reductive, $\mathrm{ssrank}(\overline{G}_\alpha) = 1$.

$\chi(R_\alpha) = 1 \Rightarrow F|_{G_\alpha} \in k[G_\alpha]^{R_\alpha} = k[\overline{G}_\alpha]$.

$B_\alpha \subseteq B \Rightarrow F(gb) = \chi(b) F(g) \ \forall g \in \overline{G}_\alpha, b \in \overline{B}_\alpha$.

$F(n_\alpha) \neq F(e) \Rightarrow F|_{G_\alpha}$ not constant on $(\overline{G}_\alpha, \overline{G}_\alpha)$.

$\therefore \langle \chi, \alpha^\vee \rangle > 0$.

$\square$

Two roots

$(X, R, X^\vee, R^\vee)$ root datum.

Assume $\alpha, \beta \in R$ not parallel.

$\sigma = s_\alpha s_\beta \in GL(V)$, $V = \mathrm{Span}_{\mathbb{R}}\{\alpha, \beta\}$.

$\sigma \neq 1$ and $\det(\sigma) = 1 \Rightarrow \chi_\sigma(1) \neq 0$.

$a = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$.

<u>Prop</u> $a \in \{0, 1, 2, 3\}$, $\mathrm{ord}(\sigma) \in \{2, 3, 4, 6\}$,
and $\langle \alpha, \beta^\vee \rangle = 0 \Leftrightarrow \langle \beta, \alpha^\vee \rangle = 0$.

<u>Proof</u>

$$\begin{aligned} \sigma.\alpha &= s_\alpha.(\alpha - \langle \alpha, \beta^\vee \rangle \beta) &&= (a-1)\alpha - \langle \alpha, \beta^\vee \rangle \beta \\ \sigma.\beta &= -s_\alpha.\beta &&= \langle \beta, \alpha^\vee \rangle \alpha - \beta \end{aligned}$$

$\chi_\sigma(\lambda) = (\lambda + 1 - a)(\lambda + 1) + a = (\lambda + 1)^2 - a(\lambda + 1) + a$.

$\chi_\sigma(\lambda) = 0 \Leftrightarrow \lambda = \frac{1}{2}a - 1 \pm \frac{1}{2}\sqrt{a^2 - 4a}$.

$\sigma \in W$ has finite order $\Rightarrow$ σ diagonalizable $/\mathbb{C}$.

Eigenvalues are roots of unity $\neq 1$.

$a < 0$ or $a \geq 4$: Real eigenvalue $\neq -1$. $\lightning$

$a = 0$: $\lambda = -1$, $\sigma = -1$. $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$.

$a = 1$: $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. $\mathrm{ord}(\sigma) = 3$.

$a = 2$: $\lambda = \pm\sqrt{-1}$ $\mathrm{ord}(\sigma) = 4$.

$a = 3$: $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. $\mathrm{ord}(\sigma) = 6$.

$\square$