LAG 26 2026-04-23

Assume $\alpha, \beta \in R$ not parallel.

Cor $\langle \alpha, \beta^\vee \rangle < 0 \Rightarrow \alpha + \beta \in R$

$\langle \alpha, \beta^\vee \rangle > 0 \Rightarrow \alpha - \beta \in R$

Proof

$\langle \alpha, \beta^\vee \rangle = -1 \Rightarrow \alpha + \beta = s_\beta.\alpha \in R.$

$\langle \beta, \alpha^\vee \rangle = -1 \Rightarrow \alpha + \beta = s_\alpha.\beta \in R.$

$\langle \alpha, \beta^\vee \rangle > 0 \Rightarrow \langle -\alpha, \beta^\vee \rangle < 0 \Rightarrow -\alpha + \beta \in R.$

$\square$

Let $R^+ \subseteq R$ be a choice of positive roots.

Cor $\exists\ w \in \langle s_\alpha, s_\beta \rangle :\ w.\alpha > 0$ and $w.\beta > 0$.

Proof

$\sigma = s_\alpha s_\beta.$

$\sigma^{12} = 1$ and $\chi_\sigma(1) \neq 0 \Rightarrow 1 + \sigma + \cdots + \sigma^{10} + \sigma^{11} = 0.$

WLOG: $\beta > 0$ (Else replace $\alpha \mapsto s_\beta.\alpha$, $\beta \mapsto s_\beta.\beta$.)

Choose $d \geq 0$ minimal with $\sigma^d.\alpha > 0$ or $\sigma^d s_\alpha.\beta > 0$.

$\sigma^d.\alpha < 0$: $w = \sigma^d s_\alpha$ works.

$\sigma^d.\alpha > 0$: $w = \sigma^d$ works

since $\sigma^d.\beta = \sigma^{d-1} s_\alpha.(-\beta) > 0.$

$\square$

Unipotent Radical

G connected LAG, $T \subseteq G$ max. torus.

$$C = \Big(\bigcap_{\substack{B \supset T \\ \text{Borel}}} B_u \Big)^\circ$$

Given $\alpha \in R(G,T)$, set $C_\alpha = \langle C, (B_\alpha)_u \rangle$.

<u>Lemma</u> C_α is unipotent and $\dim(C_\alpha/C) \leq 1$.

<u>Proof</u>

T normalizes C and C_α.

$L(C) = \oplus L(C)_\gamma \subseteq L(C_\alpha) = \oplus L(C_\alpha)_\gamma, \quad \gamma \in X^*(T)$.

<u>Show</u>: $L(C)_\gamma = L(C_\alpha)_\gamma$ for $\gamma \neq \alpha$ and

$\dim L(C_\alpha)_\alpha \leq \dim L(C)_\alpha + 1$.

$\gamma = 0$: $L(G)_0 = L(G_0)$, $G_0 = Z_G(T) = T \times Z_G(T)_u$

$Z_G(T)_u \subseteq C$. $L(C)_0 = L(Z_G(T)_u) = L(C_\alpha)_0$.

Assume $\gamma \neq 0$.

<u>Note</u>: $B \supset T \Rightarrow R_\gamma \subseteq (B \cap G_\gamma)_u \subseteq B_u$.

$\therefore R_\gamma \subseteq C$.

$L(R_\gamma)_\gamma \subseteq L(C)_\gamma \subseteq L(G)_\gamma = L(G_\gamma)_\gamma$

$\dim\big(L(G)_\gamma / L(C)_\gamma\big) \leq \dim L(G_\gamma/R_\gamma)_\gamma \leq 1$.

<u>Enough</u>: $L(C)_\gamma \subsetneq L(C_\alpha)_\gamma \Rightarrow \gamma = \alpha$.

Assume $L(C)_\gamma \subsetneq L(C_\alpha)_\gamma$ and $\gamma \neq \alpha$.

Then $\gamma \in R(G,T)$ and $L(C_\alpha)_\gamma = L(G)_\gamma$.

Choose $B \supset T$.

Choose $w \in W$ s.t. $w.\alpha \in R^+(B)$, $w.\gamma \in R^-(B)$.

Replace $B \mapsto \dot{w}^{-1} B \dot{w}$.

$\alpha \in R^+(B)$, $\gamma \in R^-(B)$. $B_\alpha \subseteq B$, $B_{-\gamma} \subseteq B$.

$C_\alpha = \langle C, (B_\alpha)_u \rangle \subseteq B_u \Rightarrow C_\alpha$ unipotent.

$L(G_\gamma / R_\gamma) = L(T) \oplus k_\gamma \oplus k_{-\gamma}$

$L(B_{-\gamma} / R_\gamma) = L(T) \oplus k_{-\gamma}$

$\therefore L(C_\alpha)_\gamma \subseteq L(B)_\gamma = L(B_{-\gamma})_\gamma \subsetneq L(G_\gamma)_\gamma$. ↯

□

<u>Thm</u> $R_u(G) = \left(\bigcap_{B \supset T} B_u \right)^\circ$

<u>Proof</u>

$B \supset T \Rightarrow R_u(G) \subseteq B_u$

$\therefore R_u(G) \subseteq C = \left(\bigcap_{B \supset T} B_u \right)^\circ$.

C closed connected unipotent.

<u>Enough</u>: $C \lhd G$ normal.

G generated by G_γ for all $\gamma \in X^*(T)$.

G generated by T, C, C_α for all $\alpha \in R(G,T)$.

C_α conn. unipotent, $C \subsetneq C_\alpha$ proper closed conn.

$\Rightarrow C \subsetneq N_{C_\alpha}(C)^\circ$. (LAG 23)

$\therefore C$ normalized by C_α.

$\therefore C \lhd G$ normal.

□

Reductive groups

Assume G is reductive (and connected).

$T \subseteq G$ max. torus. $\quad R = R(G,T)$.

(1) $S \subseteq G$ subtorus $\Rightarrow Z_G(S)$ is reductive.

WLOG $S \subseteq T$. $Z = Z_G(S)$. $T \subseteq Z$ max. torus.

$$R_u(Z) = \Big(\bigcap_{T \subseteq B' \subseteq Z} B'_u\Big)^\circ$$
$$= \Big(\bigcap_{T \subseteq B \subseteq G} (Z \cap B)_u\Big)^\circ \subseteq \Big(\bigcap_{B \supset T} B_u\Big)^\circ = e.$$

(2) $Z_G(T) = T$.

$Z_G(T)$ is reductive and nilpotent.

(3) $Z(G) \subseteq T$.

(4) $R = \{\alpha \in X^*(T) \mid \alpha \neq 0 \text{ and } L(G)_\alpha \neq 0\}$

$\alpha \in \text{RHS} \Rightarrow G_\alpha = Z_G(T_\alpha)$ is reductive.

$L(G_\alpha)_\alpha = L(G)_\alpha \neq 0.$

(5) $\alpha \in R \Rightarrow \dim L(G)_\alpha = 1.$

(6) $L(G) = L(T) \oplus \bigoplus_{\alpha \in R} L(G)_\alpha$

$L(B) = L(T) \oplus \bigoplus_{\alpha \in R^+} L(G)_\alpha$, $\quad B \supset T$ Borel.

(7) $\dim(G) = \dim(T) + |R|.$

$\dim(B) = \dim(T) + \frac{1}{2}|R|.$

Connected LAG

G any connected LAG. $T \subseteq G$ max. torus.

$\pi : G \longrightarrow \bar{G} = G/R_u(G)$ reductive quotient.

Identify $T = \pi(T)$, max. torus in $\bar{G}$.

(1) $U \lhd G$ closed unipotent normal $\Rightarrow$ $U \subseteq R_u(G)$

$U^\circ \subseteq R_u(G)$.

$\pi(U) \subseteq \bar{G}$ finite normal unipotent.

$\pi(U) \subseteq Z(\bar{G}) \subseteq T$.

$\therefore \ \pi(U) = e$.

Let $S \subseteq G$ be a subtorus. Set $Z = Z_G(S)$.

(2) $R_u(Z) = Z \cap R_u(G)$.

WLOG $S \subseteq T$.

$R_u(G) \lhd G$ normal

$\Rightarrow Z \cap R_u(G) \lhd Z$ normal & unipotent.

$\Rightarrow Z \cap R_u(G) \subseteq R_u(Z)$.

$$R_u(Z) = \Big(\bigcap_{T \subseteq B' \subseteq Z} B'_u\Big)^\circ$$

$$= \Big(\bigcap_{T \subseteq B \subseteq G} (Z \cap B)_u\Big)^\circ \subseteq R_u(G).$$

(3) $Z/R_u(Z) \xrightarrow{\subseteq} Z_{\bar{G}}(S)$ is injective.

(4) $\alpha \in X^*(T) \Rightarrow G_\alpha/R_\alpha \xrightarrow{\cong} \overline{G}_\alpha.$

$G_\alpha/R_\alpha \xrightarrow{\subseteq} \overline{G}_\alpha.$

Both groups are reductive of ssrank ≤ 1.

Same max. torus T.

$\mathrm{ssrank}(G_\alpha/R_\alpha) = 1 \Rightarrow \mathrm{ssrank}(\overline{G}_\alpha) = 1.$

Assume $\mathrm{ssrank}(\overline{G}_\alpha) = 1$.

$s_\alpha \in W(\overline{G}_\alpha, T) \subseteq W(\overline{G}, T) = W(G,T).$

$\dot{s}_\alpha \in N_G(T)$ representative.

$T_\alpha \subseteq Z(\overline{G}_\alpha) \Rightarrow s_\alpha$ identity on T_α

$\Rightarrow \dot{s}_\alpha \in Z_G(T_\alpha) = G_\alpha.$

$\therefore\ s_\alpha \in W(G_\alpha, T) = W(G_\alpha/R_\alpha, T)$

$\Rightarrow \mathrm{ssrank}(G_\alpha/R_\alpha) = 1$

$L(T) \xrightarrow{\cong} L(\pi(T)),\quad L(G_\alpha/R_\alpha)_\alpha \xrightarrow{\cong} L(\overline{G}_\alpha)_\alpha.$

(5) $R(G,T) = R(\overline{G},T).$

$L(G_\alpha/R_\alpha)_\alpha \neq 0 \Leftrightarrow L(\overline{G}_\alpha)_\alpha \neq 0.$

(3^+) $Z/R_u(Z) \xrightarrow{\cong} Z_{\overline{G}}(S)$ isomorphism.

$R(Z,T) = \{\alpha \in R(G,T) \mid \alpha(S) = 1\} = R(Z_{\overline{G}}(S), T)$

$\dim(Z/R_u(Z)) = \dim(T) + |R(Z,T)| = \dim(Z_{\overline{G}}(S)).$