LAG 27 2026-04-28

From now: G reductive (connected) LAG. $T \subseteq G$ max. torus.

$R = R(G,T)$ root system.

$W = W(G,T) = N_G(T)/T$ Weil group.

Given $\alpha \in R$: $T_\alpha = \mathrm{Ker}(\alpha)^\circ$, $G_\alpha = Z_G(T_\alpha)$.

$B_\alpha \subseteq G_\alpha$ unique Borel s.t. $T \subseteq B_\alpha$ and $L(B_\alpha)_\alpha \neq 0$.

$U_\alpha = (B_\alpha)_u \cong \mathbb{G}_a$.

Note: $G = \langle T, U_\alpha : \alpha \in R \rangle$

$$L(G) = L(T) \oplus \Big(\bigoplus_{\alpha \in R} L(U_\alpha) \Big)$$

Note: $(G_\alpha, G_\alpha) = \langle U_\alpha, U_{-\alpha} \rangle$.

$H = \langle U_{\pm\alpha} \rangle \subseteq (G_\alpha, G_\alpha)$ by LAG 24.

$H \subsetneq (G_\alpha, G_\alpha) \Rightarrow \dim H = 2$

$\Rightarrow H \subseteq (G_\alpha, G_\alpha)$ Borel $\Rightarrow H_u \cong \mathbb{G}_a$ ↯

Thm G semi-simple $\Rightarrow G = (G,G) = \langle U_\alpha : \alpha \in R \rangle$

Proof

$$H = \Big(\bigcap_{\alpha \in R} \mathrm{Ker}(\alpha) \Big)^\circ \subseteq T.$$

$H \subseteq T_\alpha \subseteq Z(G_\alpha)$ for all $\alpha \in R$ $\Rightarrow H \subseteq Z(G)^\circ = e$.

$\Rightarrow \mathrm{Span}_{\mathbb{R}}(R) = X^*(T)_{\mathbb{R}} \Rightarrow T = \langle \alpha^\vee(\mathbb{G}_m) : \alpha \in R \rangle$.

$\alpha^\vee(\mathbb{G}_m) \subseteq (G_\alpha, G_\alpha) = \langle U_{\pm\alpha} \rangle$.

$\therefore G = \langle T, U_\alpha : \alpha \in R \rangle = \langle U_\alpha : \alpha \in R \rangle$.

$U_\alpha \subseteq (G_\alpha, G_\alpha) \subseteq (G,G) \Rightarrow G = (G,G)$.

$\square$

Cor G reductive.

(1) $G = R(G)\,(G,G)$.

(2) (G,G) is semi-simple.

(3) $(G,G) \twoheadrightarrow G/R(G)$ is finite and surjective.

Proof

$\bar{G} = G/R(G)$ semi-simple $\Rightarrow \bar{G} = (\bar{G},\bar{G}) \Rightarrow G = R(G)\,(G,G)$.

Recall: $R(G) \cap (G,G)$ is finite.

$U_\alpha \subseteq (G_\alpha, G_\alpha) \subseteq (G,G) \quad \forall \alpha \in R$.

$\therefore$ $(G,G) \twoheadrightarrow G/R(G)$ finite & surjective.

$R((G,G)) \longmapsto e \;\Rightarrow\; R((G,G)) = e$.

$\square$

Prop. 1 H connected solvable LAG, $S \subseteq H$ max. torus, $U_1, \dots, U_n \subseteq H$ closed subgps. Assume:

(1) $U_i \cong \mathbb{G}_a$

(2) $L(H_u) = L(U_1) \oplus \cdots \oplus L(U_n)$.

(3) $S \subseteq N_G(U_i)$.

(4) $L(U_i) = L(H)_{\beta_i}$, $\beta_i \in X^*(S)$.

(5) $i \neq j \Rightarrow \beta_i \notin \mathbb{Q}\beta_j$.

Then $U_1 \times U_2 \times \cdots \times U_n \xrightarrow{\cong} H_u$ iso. of varieties.

$$(u_1, u_2, \dots, u_n) \longmapsto u_1 u_2 \cdots u_n.$$

Proof

Show: $S \times U_1 \times \cdots \times U_n \xrightarrow{\cong} H$

WLOG $n \geq 1$.

$\exists$ closed normal $N \triangleleft H$ s.t. $N \cong \mathbb{G}_a$ and $N \subseteq Z(H_u)$ (LAG 18).

$L(N) \subseteq L(H_u)$ S-stable subspace.

$L(N) = L(U_j)$ for some j.

$H' = Z_H(\mathrm{Ker}(\beta_j)^\circ)$.

$L(H') = \bigoplus_{\gamma \in \mathbb{Q}\beta_j} L(H)_\gamma = L(S) \oplus L(U_j)$.

$S \ltimes N = H' = S \ltimes U_j$.

$\therefore$ $N = H'_u = U_j$.

WLOG: $N = U_n$ (since $N \subseteq Z(H_u)$.)

$\pi : H \longrightarrow H/U_n$. $\mathrm{Ker}(d\pi) = L(U_n)$.

Note: $U_i \cap U_n = e$ for $i \neq n$.

$U_i \xrightarrow{\cong} \pi(U_i)$, $S \xrightarrow{\cong} \pi(S)$.

Induction on $n \Rightarrow S \times U_1 \times \cdots \times U_{n-1} \xrightarrow{\cong} H/U_n$.

$\psi : S \times U_1 \times \cdots \times U_n \longrightarrow H$ bijective.

$d\psi_{(e,e,\ldots,e)}$ bijective $\Rightarrow$ ψ birational.

Zariski $\Rightarrow$ ψ isomorphism.

$\square$

Fix total order on R.

Cor $T \subseteq B \subseteq G$, B Borel $\Rightarrow \prod_{\alpha \in R^+(B)} U_\alpha \xrightarrow{\cong} B_u$.

Choose $u_\alpha : \mathbb{G}_a \xrightarrow{\cong} U_\alpha$ for each $\alpha \in R$.

Prop. 2 Let $\alpha, \beta \in R$, $\alpha \neq \pm\beta$. $\exists$ constants $c_{ij} \in k$:

$$(u_\alpha(x), u_\beta(y)) = \prod_{\substack{i\alpha + j\beta \in R \\ i,j>0}} u_{i\alpha+j\beta}(c_{ij}\, x^i y^j)$$

Proof

WLOG: $\alpha, \beta \in R^+ = R^+(B)$, $T \subseteq B \subseteq G$.

$$(u_\alpha(x), u_\beta(y)) \in B_u \Rightarrow (u_\alpha(x), u_\beta(y)) = \prod_{\gamma \in R^+} u_\gamma(P_\gamma(x,y)).$$

$$P_\gamma(x,y) = \sum_{i,j \geq 0} c^\gamma_{ij}\, x^i y^j \in k[x,y].$$

$$t\,(u_\alpha(x), u_\beta(y))\,t^{-1} = \prod_{\gamma \in R^+} t\, u_\gamma(P_\gamma(x,y))\, t^{-1}$$

$$\Rightarrow (u_\alpha(\alpha(t)x), u_\beta(\beta(t)y)) = \prod_{\gamma \in R^+} u_\gamma(\gamma(t)\, P_\gamma(x,y)).$$

$$\therefore \gamma(t)\, P_\gamma(x,y) = P_\gamma(\alpha(t)x, \beta(t)y) = \sum_{i,j \geq 0} c^\gamma_{ij}\, (\alpha(t)x)^i\, (\beta(t)y)^j$$

$$= \sum_{i,j \geq 0} (c^\gamma_{ij}\, x^i y^j)\, (i\alpha + j\beta)(t)$$

Linear independence of characters:

$$c^\gamma_{ij} \neq 0 \Rightarrow \gamma = i\alpha + j\beta, \quad P_\gamma(x,y) = c^\gamma_{ij}\, x^i y^j.$$

Note: $e = (u_\alpha(x), u_\beta(0)) = u_\alpha(c^\alpha_{10}\, x)$.

$\therefore c^\alpha_{10} = 0$.

Symmetry: $c^\beta_{01} = 0$.

□

Choose system of positive roots $R^+ \subseteq R$.

Def The *height* of $\alpha \in R^+$ is

$$\mathrm{ht}(\alpha) = \max\{ n \in \mathbb{N} \mid \alpha \in \underbrace{R^+ + R^+ + \dots + R^+}_{n} \}.$$

Prop. 3 Assume $A \subseteq R^+$ satisfies

$$\alpha, \beta \in A \Rightarrow (\mathbb{N}\alpha + \mathbb{N}\beta) \cap R^+ \subseteq A.$$

Then $H = \langle U_\alpha : \alpha \in A \rangle$ is unipotent and

$\prod_{\alpha \in A} U_\alpha \xrightarrow{\cong} H$ iso. of varieties.

Proof

Chose $\beta \in A$ with $\mathrm{ht}(\beta)$ minimal.

$A' = A - \{\beta\}$, $H' = \langle U_\alpha : \alpha \in A' \rangle$.

Induction on $|A|$ $\Rightarrow$ H' unipotent and $\prod_{\alpha \in A'} U_\alpha \xrightarrow{\cong} H'$.

Prop. 2 $\Rightarrow$ U_β normalizes H'.

$\therefore$ $H = U_\beta \ltimes H'$.

$(H, H) \subseteq H'$ $\Rightarrow$ H solvable.

$U_\beta \subseteq H_u$ and $H_u \subseteq H$ subgroup $\Rightarrow$ $H = H_u$ unipotent.

T normalizes H $\Rightarrow$ $T \ltimes H$ solvable.

Prop. 1 $\Rightarrow$ $\prod_{\alpha \in A} \xrightarrow{\cong} H$.

□

Cor $R^+ \subseteq R$ system of positive roots

$\Rightarrow B = \langle T, U_\alpha : \alpha \in R^+ \rangle \subseteq G$ Borel subgroup.

Proof

$H = \langle U_\alpha : \alpha \in R^+ \rangle$ is unipotent, $T \subseteq N_G(H)$.

$B = T \ltimes H$ is solvable, $\dim(B) = \dim(T) + |R^+|$.

□

Note: $\mathcal{B}^T = \{B \supseteq T\} \longleftrightarrow \{R^+ \subseteq R \text{ system of pos roots}\}$

$$B \longmapsto R^+(B)$$

$$\langle T, U_\alpha : \alpha \in R^+ \rangle \longleftarrow\!\shortmid \; R^+$$

$W \circlearrowright \mathcal{B}^T$ simply transitive.

$R^+(w.B) = w.R^+(B)$

Def: R^+, $\tilde{R}^+$ are adjacent if $|R^+ \cap \tilde{R}^+| = |R^+| - 1$.

Lemma $R^+, \tilde{R}^+$ adjacent $\Rightarrow \exists!\ \beta \in R^+ : \tilde{R}^+ = s_\beta . R^+$

Proof

$A = R^+ \cap \tilde{R}^+$. $R^+ = A \cup \{\beta\}$, $\tilde{R}^+ = A \cup \{-\beta\}$.

Let $\alpha \in A$. $s_\beta.\alpha = \alpha - \langle \alpha, \beta^\vee \rangle \beta$.

$\langle \alpha, \beta^\vee \rangle \leq 0$: $s_\beta.\alpha \in R^+ \setminus \{\beta\} = A$.

$\langle \alpha, \beta^\vee \rangle \geq 0$: $s_\beta.\alpha \in \tilde{R}^+ \setminus \{-\beta\} = A$.

$\therefore$ $s_\beta . A = A$. $s_\beta.\beta = -\beta$.

$\square$