LAG 4 2026-01-29

Prop $a \in End(V)$, $\dim(V) < \infty$.

(1) $\exists!\, a_s, a_u \in End(V)$: a_s ss, a_u nilpot, $a_s a_u = a_u a_s$, $a = a_s + a_u$.

(2) $a_s, a_u \in k[a]$ (polys. in a).

(3) $W \subseteq V$, $a(W) \subseteq W \Rightarrow a_s(W) \subseteq W$ and $a_u(W) \subseteq W$.

$(a|_W)_s = a_s|_W$ and $(a|_W)_u = a_u|_W$.

$\bar{a} : V/W \longrightarrow V/W$. $(\bar{a})_s = \overline{a_s}$ and $(\bar{a})_u = \overline{a_u}$.

(4) $\phi : V \longrightarrow W$ linear, $b \in End(W)$.

$\phi a = b \phi \Rightarrow \phi a_s = b_s \phi$ and $\phi a_u = b_u \phi$.

Proof of (4):

$$\begin{array}{ccccc} & a & & a \oplus b & & b \\ & V & \longrightarrow & V \oplus W & \longrightarrow & W \\ & & v \mapsto (v, \phi(v)) & & & \end{array}$$

Cor $a \in GL(V)$, $\dim(V) < \infty$.

$\exists!\, a_s, a_u \in GL(V)$: a_s ss, a_u unipotent,

$a = a_s a_u = a_u a_s$

Proof $a = a_s + a_u = a_s(1 + a_s^{-1} a_u)$.

Locally finite

V any vector space, $a \in \mathrm{End}(V)$.

a is locally finite: $\forall v \in V\ \exists v \in W \subseteq V : a(W) \subseteq W,\ \dim(W) < \infty$.

Assume a locally finite.

a semi-simple: $a|_W$ simisimple $\forall W \subseteq V,\ a(W) \subseteq W,\ \dim < \infty$

a locally nilpotent: $a|_W$ nilpot. —"—

a locally unipotent: $a|_W$ unipot. —"—

Example $a = J_1(0) \oplus J_2(0) \oplus J_3(0) \oplus \cdots$ locally nilpot., not nilpot.

Cor $a \in \mathrm{End}(V)$ loc. finite $\Rightarrow$
$\exists!\ a_s, a_n \in \mathrm{End}(V)$ loc. finite: a_s ss, a_n loc. nilpot.

$$a = a_s + a_n, \quad a_s a_n = a_n a_s.$$

$$a_s|_W = (a|_W)_s, \quad a_n|_W = (a|_W)_n.$$

Cor $a \in GL(V)$ loc. finite
$\exists!\ a_s, a_u \in GL(V)$ loc. finite: a_s ss, a_u loc. unipot.

$$a = a_s a_u = a_u a_s.$$

G LAG.

Recall: $\rho(g): k[G] \longrightarrow k[G]$ locally finite $\forall g \in G$.

Lemma $G = GL(V)$, $\dim(V) < \infty$.

$g \in G$ is ss/unipot $\Leftrightarrow$ $\rho(g)$ ss / loc. unipot.

Proof

For $f \in V^*$, def. $\tilde{f}: V \longrightarrow k[G]$, $\tilde{f}(v)(g) = f(gv)$

$\tilde{f}(gv) = \rho(g)\tilde{f}(v)$:

$$\tilde{f}(gv)(h) = f(hgv) = \tilde{f}(v)(hg) = \big(\rho(g)\tilde{f}(v)\big)(h).$$

$$\rho(g)_s \tilde{f}(v) \underset{\substack{\uparrow \\ \text{Prop (4)}}}{=} \tilde{f}(g_s v) = \rho(g_s)\tilde{f}(v).$$

$k[G]$ gen. by $\{\tilde{f}(v) \mid f \in V^*, v \in V\}$

$\square$ $\Rightarrow$ $\rho(g)_s = \rho(g_s)$.

Def $g \in G$ is semi-simple if $\rho(g)$ semi-simple.

$g \in G$ is unipotent if $\rho(g)$ loc. unipot.

Thm G LAG, $g \in G$.

(1) $\exists!\, g_s, g_u \in G$: g_s ss, g_u unipot., $g = g_s g_u = g_u g_s$.

(2) $\phi: G \longrightarrow G'$ alg. group hom.

$\Rightarrow$ $\phi(g)_s = \phi(g_s)$ and $\phi(g)_u = \phi(g_u)$.

(3) $G = GL_n$ $\Rightarrow$ g_s, g_u are as above.

Recall: G simple: G solvable $\Leftrightarrow$ nilpotent $\Leftrightarrow$ abelian.

G nilpotent $\Leftrightarrow$ $G/Z(G)$ nilpotent.

$H \triangleleft G$ normal: G solvable $\Leftrightarrow$ G/H, H solvable.

G abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable.

Exer $U_n = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \subseteq B_n = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \subseteq GL_n$

U_n nilpotent; B_n solvable, not nilpotent.

Burnside's Thm

V vector space over $k = \bar{k}$, $\dim(V) < \infty$.

$R \subseteq \mathrm{End}_k(V)$ subalgebra.

V simple R-module $\Rightarrow R = \mathrm{End}_k(V)$.

Cor $G \subseteq GL(V)$ closed subgroup, $\dim(V) < \infty$.

If V irred. G-rep. then $\mathrm{End}_k(V) = \mathrm{span}(G)$.

Cor $G \subseteq GL_n$ consists of unipotent elts.

$\Rightarrow xGx^{-1} \subseteq U_n$ for some $x \in GL_n$.

Proof Enough: $xGx^{-1} \subseteq B_n$.

$V = k^n$ not irred: $\exists\ 0 \neq W \subsetneq V$, W G-stable.

Images of G in $GL(W)$, $GL(V/W)$ consist of unipot. elts.

Use induction on n.

WLOG V irreducible.

For $g, h \in G$: $\mathrm{Tr}((1-g)h) = \mathrm{Tr}(h) - \mathrm{Tr}(gh) = n - n = 0$.

Burnside $\Rightarrow \mathrm{Tr}((1-g)h) = 0 \quad \forall\, h \in \mathrm{End}(V) \Rightarrow g = e$.

$\square$

<u>Def</u> G is <u>unipotent</u> if all elts of G are unipot.

<u>Cor</u> unipotent $\Rightarrow$ nilpotent.

<u>Prop</u> G unipotent, X affine G-variety.

$\Rightarrow$ $G.x \subseteq X$ is closed $\forall x \in X$.

<u>Proof</u>

$O \subseteq X$ orbit.

WLOG: $X = \overline{O}$. $\Rightarrow$ $O \subseteq X$ is open.

$Y = X \setminus O \subseteq X$ closed, G-stable.

$s : G \longrightarrow GL(k[X])$, $(s(g)f)(x) = f(g^{-1}.x)$

G acts locally finitely on $k[X]$.

$G.I(Y) \subseteq I(Y)$.

$\exists\, 0 \neq f \in I(Y) :$ $s(g)f = f$ $\forall g \in G$:

$\exists\, 0 \neq W \subseteq I(Y)$, $\dim(W) < \infty$, $G.W \subseteq W$.

G unipotent $\Rightarrow$ $s(G) \subseteq GL(W)$ unipotent

WLOG: $s(G) \subseteq U_m \subseteq GL_m = GL(W)$.

$\Rightarrow$ f constant on O

$\Rightarrow$ f constant on X

$I(Y) \supseteq \langle f \rangle = k[X] \Rightarrow Y = \emptyset$.

$\square$

Def G LAG.

$$\left.\begin{aligned} G_s &= \{ g \in G \mid g \text{ is semi-simple} \} \\ G_u &= \{ g \in G \mid g \text{ is unipotent} \} \end{aligned}\right\} \text{ subsets, usually not subgroups.}$$

Note: $G_u \subseteq G$ is closed.

$$(GL_n)_u = \{ g \in GL_n \mid \chi_g(t) = (t-1)^n \}$$

Thm G commutative LAG.

(1) G_s and G_u are closed subgroups.

(2) $\mu : G_s \times G_u \xrightarrow{\cong} G$ (product map).

Proof

$(gh)_s = g_s h_s = gh \Rightarrow G_s$ subgroup.

WLOG: $G \subseteq B_n \subseteq GL_n$.

$G_s = G \cap D_n$, $G_u = G \cap U_n$ closed. $D_n = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$

$\mu : G_s \times G_u \longrightarrow G$, $(g, h) \mapsto gh$ is bijective

(unique Jordan decomp.)

$\mu^{-1}(g) = (g_s, g_s^{-1} g)$ is morphism of varieties.

$\square$

Cor G commutative & connected $\Rightarrow$ so are G_s, G_u.