LAG 6 2026-02-05

T torus.

$X^*(T) = \{\chi : T \to \mathbb{G}_m\}$ group of characters.

$X_*(T) = \{\lambda : \mathbb{G}_m \to T\}$ group of cocharacters.

<u>Pairing</u>: $X^*(T) \times X_*(T) \longrightarrow X^*(\mathbb{G}_m) = \mathbb{Z}$

$$(\chi, \lambda) \longmapsto \chi\lambda \longleftrightarrow \langle \chi, \lambda \rangle$$

$\chi\lambda(a) = a^{\langle \chi, \lambda \rangle}$ for $a \in \mathbb{G}_m$.

<u>Exer</u>: Perfect pairing.

$\mathbb{G}_m = k^\times = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$.

<u>Def</u> $\phi : \mathbb{G}_m \longrightarrow Z$ morphism, $z \in Z$ point.

$\lim_{a \to 0} \phi(a) = z \iff \exists$ morphism $\tilde{\phi} : \mathbb{A}^1 \longrightarrow Z$:

$\tilde{\phi}(a) = \phi(a)$ for $a \in \mathbb{G}_m$, $\tilde{\phi}(0) = z$.

$\lim_{a \to \infty} \phi(a) = z \iff \lim_{a \to 0} \phi(a^{-1}) = z$.

Always exist if Z is projective (or complete).

Assume Z affine.

$\phi^* : k[Z] \longrightarrow k[t, t^{-1}]$.

$\lim_{a \to 0} \phi(a)$ exists $\iff \phi^*(k[Z]) \subseteq k[t]$

$\iff \forall f \in k[Z] : f\phi \in k(\mathbb{A}^1)$ is defined at 0.

Def T torus, V T-variety, $\lambda \in X_*(T)$.

$$V(\lambda) = \{ v \in V \mid \lim_{a \to 0} \lambda(a).v \text{ exists} \}$$

Note: $V(-\lambda) = \{ v \in V \mid \lim_{a \to \infty} \lambda(a).v \text{ exists} \}$

Lemma T torus, V affine T-variety, $\lambda \in X_*(T)$.

(1) $V(\lambda) \subseteq V$ is closed.

(2) $V(\lambda) \cap V(-\lambda) = V^{\lambda(\mathbb{G}_m)} = \{ v \in V \mid \lambda(a).v = v \ \forall a \in \mathbb{G}_m \}$.

Proof

$T \circlearrowright k[V]$ locally finite.

$$(s(t).f)(v) = f(t^{-1}.v).$$

$$k[V] = \bigoplus_{x} k[V]_x$$

$$f = \sum_x f_x \quad \Rightarrow \quad s(t).f = \sum_x x(t) f_x .$$

Let $v \in V$.

$$\phi : \mathbb{G}_m \longrightarrow V, \quad \phi(a) = \lambda(a).v$$

$$\phi^*(f)(a) = f(\lambda(a).v) = (s(\lambda(a)^{-1}).f)(v)$$

$$= \sum_x a^{-\langle x, \lambda \rangle} f_x(v).$$

Defined at $a = 0 \iff f_x(v) = 0$ when $\langle x, \lambda \rangle > 0$.

$$V(\lambda) = Z\Big(\bigoplus_{\langle x, \lambda \rangle > 0} k[V]_x \Big) \qquad \text{(not ideal!)}$$

$$f(\lambda(a).v) = \sum_{x} a^{-\langle x,\lambda\rangle} f_x(v)$$

$$\begin{aligned} v \in V(\lambda) \cap V(-\lambda) &\Leftrightarrow \forall f \in k[V] : f_x(v) = 0 \text{ for } \langle x,\lambda\rangle \neq 0 \\ &\Leftrightarrow \forall f \in k[V] : f(\lambda(a).v) = f(v) \\ &\Leftrightarrow v \in V^{\lambda(\mathbb{G}_m)} \end{aligned}$$

$\square$

<u>Example</u>

$\mathbb{G}_m \circlearrowright \mathbb{A}^2, \quad a.(x,y) = (ax, a^{-1}y).$

$\mathbb{G}_m \circlearrowright k[\mathbb{A}^2] = k[X,Y].$

$(a.f)(x,y) = f(a^{-1}.(x,y)) = f(a^{-1}x, ay).$

$a.X = a^{-1}X, \quad a.Y = aY$

$\lambda = id : \mathbb{G}_m \longrightarrow \mathbb{G}_m.$

$(x,y) \in \mathbb{A}^2(\lambda) \Leftrightarrow \lim_{a \to 0} a.(x,y)$ exists $\Leftrightarrow Y = 0.$

$\mathbb{A}^2(\lambda) = Z(Y), \quad \mathbb{A}^2(-\lambda) = Z(X).$

$\mathbb{A}^2(\lambda) \cap \mathbb{A}^2(-\lambda) = \{(0,0)\} = (\mathbb{A}^2)^{\mathbb{G}_m}$

$k[\mathbb{A}^2]_x = \mathrm{Span}\{X^iY^j \mid j-i = x\}$

$\bigoplus_{\langle x,\lambda\rangle > 0} k[\mathbb{A}^2]_x = \mathrm{Span}\{X^iY^j \mid j-i > 0\}$ (not ideal!)

Generates $I(\mathbb{A}^2(\lambda)) = \langle Y \rangle \subseteq k[\mathbb{A}^2].$

Quiz

G LAG, $g \in G$ torsion elt. $g^m = e$.

$g = g_s g_u = ?$

$p = \mathrm{char}(k) = 0$: $\quad g = g_s$.

Assume $p > 0$:

$p \nmid m \Rightarrow g = g_s$.

$m = p^j \Rightarrow g = g_u$.

$m = n p^j$, $\quad p \nmid n$.

g^n is unipotent.

g^{p^j} is semi-simple.

$an + bp^j = 1$, $\quad a, b \in \mathbb{Z}$.

$g = (g^{bp^j})(g^{an}) = g_s g_u$.

Additive functions

G LAG. $p = \mathrm{char}(k)$.

$\mathbb{G}_a = k$ (additive group)

Def Additive functions on G:

$$\mathcal{A}(G) = \{ f \in k[G] \mid f : G \longrightarrow \mathbb{G}_a \text{ group hom.} \}$$

Example $G = \mathbb{G}_a^n$ vector group.

$k[G] = k[T_1, \dots, T_n]$

$f \in k[G]$ additive $\Leftrightarrow$ $f(xy) = f(x) + f(y)$ $\forall\, x, y \in G$

$\Leftrightarrow f(T_1 + U_1, \dots, T_n + U_n) = f(T_1, \dots, T_n) + f(U_1, \dots, U_n)$.

Claim:

$$\mathcal{A}(G) = \begin{cases} \mathrm{Span}_k\{T_1, \dots, T_n\} & \text{if } p = 0 \\ \mathrm{Span}_k\{ T_i^{p^j} \mid 1 \le i \le n,\ j \ge 0 \} & \text{if } p > 0 \end{cases}$$

Proof

$$\frac{\partial f}{\partial T_i}(T_1 + U_1, \dots, T_n + U_n) = \frac{\partial f}{\partial T_i}(T_1, \dots, T_n)$$

$$\Rightarrow \frac{\partial f}{\partial T_i}(U_1, \dots, U_n) = c_i \in k \text{ constant.}$$

$$g = f - \sum_{i=1}^{n} c_i T_i. \qquad \frac{\partial g}{\partial T_i} = 0 \quad \forall i$$

$p = 0$: $g = 0$

$p > 0$: $g = h(T_1^p, \dots, T_n^p)$, $\quad h \in k[G]$

$g \in \mathcal{A}(G) \Rightarrow h \in \mathcal{A}(G)$.

Induction on $\deg(f)$.

$\square$