# LAG 8 2026-02-12

$X$ irred. variety, $p \in X$.

$\mathcal{O}_{X,p} = \{f \in k(X) \mid f \text{ def. at } p\}$

$\mathfrak{m}_p \subseteq \mathcal{O}_{X,p}$ unique max. ideal.

$T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2$. $\quad T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p))$

<u>Exer</u>: $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = $ min. # generators of ideal $\mathfrak{m}_p$.

<u>Principal Ideal Theorem</u>:

min. # gens of $\mathfrak{m}_p \geq \dim \mathcal{O}_{X,p}$ (Krull dim.)

<u>Def</u> $\mathcal{O}_{X,p}$ is a <u>regular local ring</u> if $\mathfrak{m}_p$ is gen. by $\dim(\mathcal{O}_{X,p})$ elts.

$X$ irred. $\Rightarrow$ $\dim(X) = \dim(\mathcal{O}_{X,p})$.

$\therefore \dim_k(T_p^*X) \geq \dim(X)$.

<u>Def</u> $p \in X$ non-sing point $\Leftrightarrow$ $\mathcal{O}_{X,p}$ regular local

$\Leftrightarrow \dim_k(T_p^*X) = \dim(X)$.

<u>Theorem</u> $X_{sing} \subsetneq X$ proper closed subset.

<u>Exer</u> $X$ affine, $p \in X$. $k[X] \longrightarrow \mathcal{O}_{X,p}$ $k$-alg. hom.

$$\mathrm{Der}_k(\mathcal{O}_{X,p}, k(p)) \xrightarrow{\cong} \mathrm{Der}_k(k[X], k(p))$$

$$I(p)/I(p)^2 \xrightarrow{\cong} \mathfrak{m}_p/\mathfrak{m}_p^2$$

$$f/g(p) + I(p)^2 \longmapsfrom f/g + \mathfrak{m}_p^2$$

$f, g \in k[X]$, $f(p) = 0$, $g(p) \neq 0$.

## Differentiation:

$\phi : X \longrightarrow Y$ morphism. $\phi^* : \mathcal{O}_{Y,\phi(p)} \longrightarrow \mathcal{O}_{X,p}$

$$d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y.$$
$$D \longmapsto D\phi^*$$

$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ : $d(\psi\phi)_p = d\psi_{\phi(p)} \circ d\phi_p$

## Differentials

$R$ com. ring. $A$ com. $R$-algebra.

$\exists$ <u>universal $R$-derivation</u> $d_A : A \longrightarrow \Omega_{A/R}$ :

For any $R$-derivation $D : A \longrightarrow M$

$\exists!$ $A$-linear map $\tilde{D} : \Omega_{A/R} \longrightarrow M$ s.t. $D = \tilde{D} \circ d_A$.

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[dr, "d_A"'] & M \\ & \Omega_{A/R} \arrow[u, dashed, "\exists!\ \tilde{D}"'] \end{tikzcd}$$

## Construction:

$$\Omega_{A/R} = \left(\text{free } A\text{-module gen. by } \{d_A(b) : b \in A\}\right) \Big/ \left\langle \begin{array}{l|l} d_A(a+b) = d_A(a) + d_A(b) & a, b \in A \\ d_A(ab) = a\, d_A(b) + b\, d_A(a) & r \in R \\ d_A(r) = 0 & \end{array} \right\rangle$$

$X$ affine variety, $p \in X$.

<u>Notation</u>: $M$ $k[X]$-module.

$$M(p) = M/I(p)M = M \otimes_{k[X]} k(p).$$

$$M \longrightarrow M(p), \quad m \mapsto m(p) = m + I(p)M.$$

<u>Exer</u> $\mathrm{Hom}_{k[X]}(M, k(p)) = \mathrm{Hom}_k(M(p), k)$

$$M \longrightarrow M(p) \longrightarrow k(p).$$

<u>Def</u>: $\Omega_X = \Omega_{k[X]/k} = \{\text{covector fields on } X\}$

$d = d_X : k[X] \longrightarrow \Omega_X$ universal $k$-derivation.

$$T_pX = \mathrm{Der}_k(k[X], k(p)) = \mathrm{Hom}_{k[X]}(\Omega_X, k(p))$$
$$= \mathrm{Hom}_k(\Omega_X(p), k) = \Omega_X(p)^*$$

$$\therefore \quad \Omega_X(p) = T_p^*X = m_p/m_p^2$$
$$df(p) \longleftrightarrow f - f(p) + m_p^2.$$

<u>Exer</u> $k[\mathbb{A}^n] = k[T_1, \ldots, T_n]$.

$\Omega_{\mathbb{A}^n} = \mathrm{Span}_{k[\mathbb{A}^n]}\{dT_1, \ldots, dT_n\}$ (free!)

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i} dT_i$$

$$T_p^*\mathbb{A}^n = \mathrm{Span}_k\{dT_1, \ldots, dT_n\}$$

$$df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(p)\, dT_i$$

<u>Exer</u> $X \subseteq \mathbb{A}^n$ closed. $I(X) = \langle f_1, .., f_m \rangle \subseteq k[\mathbb{A}^n]$.

$t_i = \overline{T_i} \in k[X] = k[\mathbb{A}^n]/I(X)$.

$$\Omega_X = \mathrm{Span}_{k[X]}\{dt_1, .., dt_n\} / \langle \overline{df_1}, .., \overline{df_m} \rangle$$

$$= \left(\Omega_{\mathbb{A}^n} / \langle df_1, .., df_m \rangle\right) \otimes_{k[\mathbb{A}^n]} k[X].$$

Notation: $\overline{df} = \sum_{i=1}^{n} \overline{\frac{\partial f}{\partial T_i}} dt_i \in \mathrm{Span}_{k[X]}\{dt_1, .., dt_n\}$

$$T_p^* X = T_p^* \mathbb{A}^n / \langle df_1(p), .., df_m(p) \rangle$$

$$T_p X = \langle df_1(p), .., df_m(p) \rangle^{\perp} \subseteq T_p \mathbb{A}^n$$

<u>Jacobi matrix</u>: $J = \left(\frac{\partial f_i}{\partial T_j}\right) \in \mathrm{Mat}(n \times m, k[X])$

$$k[X]^{\oplus m} \xrightarrow{J} k[X]^{\oplus n} \longrightarrow \Omega_X \longrightarrow 0$$

$$k^{\oplus m} \xrightarrow{J(p)} k^{\oplus n} \longrightarrow T_p^* X \longrightarrow 0$$

$\therefore$ rank $J(p) \leq n - \dim(X)$

Equality $\Leftrightarrow$ $p \in X$ nonsing. point.

<u>Cor</u> $X_{sing} = \{ \mathrm{rank}(J) < \mathrm{codim}(X, \mathbb{A}^n) \} \subseteq X$ closed.

<u>Vector fields</u>: $\mathrm{Der}_k(k[X], k[X]) = \mathrm{Hom}_{k[X]}(\Omega_X, k[X])$

$X$ non-singular $\Rightarrow$ $\Omega_X$ locally free $k[X]$-module

$\Rightarrow \mathrm{Hom}_{k[X]}(\Omega_X, k[X])(p) = \mathrm{Hom}_k(\Omega_X(p), k) = T_p X$.

# Separable field extensions

$E/F$ field extension. $(F \subseteq E)$ $p = \mathrm{char}(F)$.

<u>Def</u>: $E/F$ is <u>separably algebraic</u> if

$\forall a \in E \ \exists f \in F[T] : f(a) = 0$ and $f$ has no multible roots.

<u>Note</u>: $b \in E$ is a multiple root $\Leftrightarrow f(b) = f'(b) = 0$.

$p = 0 \Rightarrow E/F$ always separable.

WLOG: $f \in F[T]$ irred.

$$f(b) = f'(b) = 0 \Rightarrow f'(T) = 0 \in F[T].$$

$$p = 0 \Rightarrow f'(T) \neq 0.$$

<u>Def</u>: Transcendence basis of $E/F$:

$B \subseteq E$ such that $B$ is alg. indep. $/F$

and $E/F(B)$ is algebraic.

$\mathrm{tr.deg}_F(E) = \# B$

<u>Def</u> $E/F$ is <u>separably generated</u> if $\exists$ tr. basis $B$

such that $E/F(B)$ is separably algebraic.

<u>Def</u> $F$ is <u>perfect</u> if $p = 0$ or $\forall r \in F \ \exists s \in F : s^p = r$.

alg. closed $\Rightarrow$ perfect.

<u>Theorem</u> $F$ perfect $\Rightarrow E/F$ separably generated.