LAG 9 2026-02-17

Rational functions

X irred. variety.

$k(X) = \{(U,f) \mid \emptyset \neq U \subseteq X,\ f: U \to k \text{ regular}\}/\sim$

$\quad = \{f : X \dashrightarrow k\}$ field of rat. fcns. on X.

X affine $\Rightarrow$ $k(X) = K(k[X])$ field of fractions.

$\phi : X \longrightarrow Y$ morphism of irred. varieties.

<u>Def</u> ϕ is <u>dominant</u> if $\overline{\phi(X)} = Y$.

Assume $\phi : X \longrightarrow Y$ dominant.

$\quad \phi^* : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$ is injective.

$\quad \phi^* : k(Y) \longrightarrow k(X)$, $\phi^* f = f\phi : X \to Y \dashrightarrow k$

<u>Def</u>: ϕ is <u>separable</u> if $k(X)/k(Y)$ is separably generated.

<u>Thm</u> $\phi : X \longrightarrow Y$ morphism of irred. varieties.

(1) Assume $p \in X$ is non-sing, $\phi(p) \in Y$ is non-sing., and $d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y$ is surjective.
Then ϕ is dominant and separable.

(2) Assume ϕ is dominant and separable.
Then assumption of (1) holds for all points p in dense open $\subseteq X$.

Let G be a connected alg. group.

<u>Cor</u> Any homogeneous G-variety X is irred. and non-singular.

<u>Cor</u> $\phi : X \longrightarrow Y$ equivariant morphism of homogeneous G-varieties. TFAE:

(1) ϕ is separable.

(2) $d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y$ is surjective for some $p \in X$.

(3) $d\phi_p$ is surjective for all $p \in X$.

<u>Cor</u> $\phi : G \longrightarrow G'$ surjective homomorphism of alg. groups. ϕ separable $\Leftrightarrow$ $d\phi_e$ surjective.

<u>Tangent spaces</u>

X affine variety, $p \in X$.

$k(p) = k[X]/I(p)$.

$d_X : k[X] \longrightarrow \Omega_X = \Omega_{k[X]/k}$.

$\Omega_X \longrightarrow \Omega_X(p) = T_p^* X = I(p)/I(p)^2$

$d_X f \longmapsto d_X f(p) \longleftrightarrow f - f(p) + I(p)^2$

$T_p X = \mathrm{Der}_k(k[X], k(p))$

Perfect pairing: $T_p^* X \times T_p X \longrightarrow k$

$(f + I(p)^2, D) \longmapsto D(f)$

Differentiation

$\phi : X \longrightarrow Y$ morphism of affine varieties, $p \in X$.

$$\begin{tikzcd}
k[Y] \arrow[r, "\phi^*"] \arrow[d, "d_Y"'] & k[X] \arrow[d, "d_X"] \\
\Omega_Y \arrow[r, "\phi^*"] \arrow[d] & \Omega_X \arrow[d] \\
T^*_{\phi(p)} Y \arrow[r, "\phi^*"] & T^*_p X
\end{tikzcd}$$

$$\phi^*(d_Y(f)) = d_X(\phi^* f)$$

$$\phi^*\left(f + I(\phi(p))^2\right) = \phi^*(f) + I(p)^2.$$

$$\phi^*\left(d_Y f(\phi(p))\right) = d_X(\phi^* f)(p)$$

$$d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y$$
$$D \longmapsto D\phi^*$$

$D \in T_p X$, $u \in T^*_{\phi(p)} Y \Rightarrow (u, d\phi_p D) = (\phi^* u, D)$

Products

Let $(p,q) \in X \times Y$

$D \in T_{(p,q)}(X \times Y) = \mathrm{Der}_k(k[X] \otimes k[Y], k(p,q))$

$D(f \otimes g) = g(q) D(f \otimes 1) + f(p) D(1 \otimes g)$.

$j_q : X \longrightarrow X \times Y$, $j_p : Y \longrightarrow X \times Y$

$T_{(p,q)}(X \times Y) = T_p X \oplus T_q Y = dj_q(T_p X) \oplus dj_p(T_q Y)$

Lemma G alg. group. $X, Y \in T_eG$.

$\mu : G \times G \longrightarrow G$ mult. $i : G \longrightarrow G$ inverse.

$d\mu_{(e,e)} : T_eG \oplus T_eG \longrightarrow T_eG$, $(X,Y) \mapsto X+Y$

$di_e : T_eG \longrightarrow T_eG$, $X \mapsto -X$.

Proof

$$G \xrightarrow{j_1} G \times G \xrightarrow{\mu} G$$
$$x \longmapsto (x,e) \longmapsto x$$

$$d\mu(X,0) = d\mu(dj_1(X)) = d(\mu j_1)(X) = X.$$

$$G \xrightarrow{\phi} G \times G \xrightarrow{\mu} G$$
$$x \longmapsto (x, x^{-1}) \longmapsto e$$

$$T_eG \xrightarrow{d\phi} T_eG \oplus T_eG \xrightarrow{d\mu} T_eG$$
$$X \longmapsto (X, di_e(X)) \longmapsto X + di_e(X) = 0.$$

$\square$

Adjoint Representation

G LAG.

$$\hat{\lambda}, \hat{\rho} : G \longrightarrow \mathrm{Aut}_{var}(G)$$

$$\hat{\lambda}(x)(y) = xy, \quad \hat{\rho}(x)(y) = yx^{-1}.$$

$$\lambda, \rho : G \longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[G])$$

$$\lambda(x) = \hat{\lambda}(x^{-1})^*, \quad \rho(x) = \hat{\rho}(x^{-1})^*$$

$$(\lambda(x)f)(y) = f(x^{-1}y), \quad (\rho(x)f)(y) = f(yx).$$

<u>Note</u>: $\lambda(x)\rho(y) = \rho(y)\lambda(x) \quad \forall\, x, y \in G.$

$$\lambda(x) = \hat{\lambda}(x^{-1})^* : T^*_{x^{-1}y}G \longrightarrow T^*_y G$$

$$\rho(x) = \hat{\rho}(x^{-1})^* : T^*_{yx} G \longrightarrow T^*_y G$$

$$\lambda(x).df(x^{-1}y) = d(\lambda(x).f)(y).$$

$$\rho(x).df(yx) = d(\rho(x).f)(y).$$

$$\mathrm{Int} : G \longrightarrow \mathrm{Aut}(G)$$

$$\mathrm{Int}(x) = \hat{\lambda}(x)\hat{\rho}(x). \quad \mathrm{Int}(x)(y) = xyx^{-1}.$$

$$\mathrm{Int}(x)^* = \lambda(x^{-1})\rho(x^{-1}) : k[G] \longrightarrow k[G]$$

$$(\mathrm{Int}(x)^*.f)(y) = f(xyx^{-1}).$$

$$\mathrm{Ad} : G \longrightarrow GL(T_e G)$$

$$\mathrm{Ad}(x) = d\,\mathrm{Int}(x)_e$$

$$\mathrm{Ad}(x).X = X\,\mathrm{Int}(x)^* = X\,\lambda(x^{-1})\rho(x^{-1}).$$