Solutions to proof problems from Homework 9

5.1(63). Prove that if v is an eigenvector of a matrix A, then for any nonzero scalar c, cv is also an eigenvector of A.

Assume that v is an eigenvector of A. Then $v \neq 0$, and we have $Av = \lambda v$ for some scalar $\lambda \in \mathbb{R}$. Let $c \in \mathbb{R}$ be any non-zero scalar. Then cv is a non-zero vector, and we have $A(cv) = cAv = c\lambda v = \lambda(cv)$. This shows that cv is an eigenvector of A (with eigenvalue λ).

5.1: 66. Prove that a square matrix is invertible if and only if 0 is not an eigenvalue.

Let A be an $n \times n$ matrix. Then $\lambda = 0$ is an eigenvalue of A if and only if there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v = 0$. In other words, 0 is an eigenvalue of A if and only if the vector equation $Ax = 0$ has a non-zero solution $x \in \mathbb{R}^n$. On the other hand, it follows from Theorem 2.6 that A is invertible if an only if the zero vector is the only solution of $Ax = 0$. We deduce that A is invertible if and only if 0 is not an eigenvalue.

5.1: 72. An $n \times n$ matrix A is called nilpotent if, for some positive integer k, $A^k = 0$, where 0 is the $n \times n$ zero matrix. Prove that 0 is the only eigenvalue of a nilpotent matrix.

Let A be a nilpotent $n \times n$ matrix and choose $k > 0$ such that $A^k = 0$. Let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Then there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. This implies that $A^r v = \lambda^r v$ for each positive integer r. For example, $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$. By taking $r = k$, we obtain $\lambda^k v = A^k v = 0$, that is $\lambda^k v$ is the zero vector. Since v is a non-zero vector, this implies that $\lambda^k = 0$. We deduce that $\lambda = 0$.

5.2: 79. Hello

Show that if A is an upper triangular or lower triangular matrix, then λ is an eigenvalue of A with multiplicity k if and only if λ appears exactly k times on the diagonal of A.

Let A be an upper (or lower) triangular $n \times n$ matrix and let $\mu_1, \mu_2, \ldots, \mu_n$ be the diagonal entries of A. Then the characteristic polynomial of A is $\chi_A(x) = \det(A - xI_n) = (\mu_1 - x)(\mu_2 - x)\cdots(\mu_n - x)$. Assume that exactly k of the eigenvalues μ_i are equal to λ, and let $g(x)$ be the product of the factors $(\mu_i - x)$ for which $\mu_i \neq \lambda$. Then $\chi_A(t) = (\lambda - x)^k g(x)$, and since $g(\lambda)$ is a product of factors $(\mu_i - \lambda) \neq 0$, we have that $g(\lambda) \neq 0$. It follows that λ is an eigenvalue with (algebraic) multiplicity k.

5.3: 81. If A is a diagonalizable matrix, prove that A^T is diagonalizable.

Assume that A is a diagonalizable matrix. Then we can find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Since P is invertible, we know that P^T is also invertible, with inverse $(P^T)^{-1} = (P^{-1})^T$. Set $Q = (P^T)^{-1} = (P^{-1})^T$. Since $D^T = D$, we obtain $QDQ^{-1} = (P^{-1})^T D^T P^T = (PDP^{-1})^T = A^T$. This shows that A^T is diagonalizable.