The Branch and Bound method for (mixed) Integer Programs

Problem P:
Maximize $z = c^T x$ subject to
$Ax = b$
$x \geq 0$ in $\mathbb{R}^s$.
$x \in \mathbb{Z}$ for $j \in I$.
Here A is an $m \times s$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^s$, and $I \subset \{1, 2, \ldots, s\}$.
Feasible solutions: $S = \{x \in \mathbb{R}^s \mid Ax = b \text{ and } x \geq 0 \text{ and } x_j \in \mathbb{Z} \text{ for } j \in I\}$

Corresponding linear problem:
Problem $\widehat{P}$:
Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$ in $\mathbb{R}^s$.
Feasible solutions: $\widehat{S} = \{x \in \mathbb{R}^s \mid Ax = b \text{ and } x \geq 0\}$
Notice that $S \subset \widehat{S}$.

Let $\widehat{x} \in \widehat{S}$ be an optimal solution to the linear problem $\widehat{P}$.
This solution $\widehat{x}$ can be found efficiently using the simplex algorithm.

Observation 1:
If $\widehat{x} \in S$, then $\widehat{x}$ is an optimal solution the the mixed integer problem P.

Assume that $\widehat{x} \notin S$.
Observation 2:
If $\widetilde{x} \in S$ is any feasible solution to P, then $\widetilde{x} \in \widehat{S}$, so we have $z(\widetilde{x}) \leq z(\widehat{x})$.
Therefore $z(\widehat{x})$ is an **upper bound** for the optimal objective function value of P.

Idea of Branch and Bound:
Assume that $\widehat{x}_k \notin \mathbb{Z}$ for some $k \in I$.
The optimal solution $\widetilde{x} \in S$ to P must satisfy: $\widetilde{x}_k \leq [\widehat{x}_k]$ **or** $\widetilde{x}_k \geq [\widehat{x}_k] + 1$.
Now solve two mixed integer problems:
P_1: Maximize z subject to $Ax = b$, $x \geq 0$, $x_j \in \mathbb{Z}$ for $j \in I$, **and** $x_k \leq [\widehat{x}_k]$.
P_2: Maximize z subject to $Ax = b$, $x \geq 0$, $x_j \in \mathbb{Z}$ for $j \in I$, **and** $x_k \geq [\widehat{x}_k] + 1$.
The optimal solution to P is the solution with the largest objective value.

Problem: Solving P_1 and P_2 in the same way can lead to lots of branching.

P

P_1 P_2

P_1' P_1'' P_2' P_2''

... ...

We must try to eliminate as many branches of the tree as possible.

Idea 2:

Start by computing optimal solutions $\widehat{x}_1$ and $\widehat{x}_2$ to the linear problems $\widehat{P}_1$ and $\widehat{P}_2$ corresponding to P_1 and P_2.

Suppose we have found some feasible solution $\widetilde{x}_1$ to P_1 such that $z(\widetilde{x}_1) \geq z(\widehat{x}_2)$.

Then by Observation 2 we know that $\widetilde{x}_1$ is as least as good as the optimal solution to P_2. So we don't have to solve P_2 after all! We say that P_2 has been **implicitly enumerated**.

Rooted trees:

A **tree** is a connected graph without loops, consisting of **nodes** connected by line segments called **branches**. A **rooted tree** is a tree together with a special node called the **root** of the tree. We will draw rooted trees so that they grow downward, i.e. the root is placed at the top and all branches go down. The nodes at the bottom, for which no branches continue further down, are called **leaves**.

Branch and Bound Algorithm:

Find an optimal solution $\widehat{x}$ to $\widehat{P}$, and let $(\widehat{x}, z(\widehat{x}))$ be the root of a tree. We will gradually build a tree under this root.

Each branch will be labeled by $x_k \leq a$ or $x_k \geq a+1$ for some $k \in I$ and $a \in \mathbb{Z}$.

Each node of the tree represents the linear problem obtained from $\widehat{P}$ by adding the additional constraints given by the branches leading to the node.

A node is labeled by either $(\widehat{x}, z(\widehat{x}))$, if $\widehat{x}$ is an optimal solution of the corresponding linear problem, or by $\emptyset$ if this linear problem has no feasible solutions.

A leaf of the tree is **terminal** if it is labeled by $\emptyset$ or by a feasible solution to the mixed integer problem P. Otherwise the leaf is called **dangling**.

A dangling leaf $(\widehat{x}, z(\widehat{x}))$ is implicitly enumerated if there exists a terminal leaf $(\widetilde{x}, z(\widetilde{x}))$ such that $z(\widehat{x}) \leq z(\widetilde{x})$.

Branching a dangling leaf:

Let $\widehat{x}$ be an optimal solution to the problem of a dangling leaf.

Choose $k \in I$ such that $\widehat{x}_k \notin \mathbb{Z}$. E.g. choose k such that $\mathrm{frac}(\widehat{x}_k)$ is closest to $\frac{1}{2}$.

Create two new branches from the leaf labeled $x_k \leq [\widehat{x}_k]$ and $x_k \geq [\widehat{x}_k]+1$, and add new leaves at the ends of these branches with the correct labels.

The algorithm:

(1) Label the root of the tree by $(\widehat{x}, z(\widehat{x}))$, where $\widehat{x}$ is an optimal solution to $\widehat{P}$.

(2) If there are dangling leaves that are not implicitly enumerated, **choose** one of them and branch it. Repeat until all dangling leaves are implicitly enumerated.

(3) The optimal solution is the terminal leaf with the largest objective value.

Example:

Maximize $z = 11x_1 + 80x_2 + 13x_3$

subject to

$3x_1 + 4x_2 + x_3 \leq 20$

$x_1 + 10x_2 + x_3 \leq 22$

$x_1 + 8x_2 + 3x_3 \leq 28$

$x \geq 0$ in $\mathbb{R}^3$

$x_1, x_3 \in \mathbb{Z}$.

$z = 208$
$x_1 = 23/7 = 3\frac{2}{7}$ $x_2 = 10/7 = 1\frac{3}{7}$ $x_3 = 31/7 = 4\frac{3}{7}$

$z = 2680/13 = 206\frac{2}{13}$
$x_1 = 44/13 = 3\frac{5}{13}$ $x_2 = 19/13 = 1\frac{6}{13}$ $x_3 = 4$

$z = 992/5 = 198\frac{2}{5}$
$x_1 = 17/5 = 3\frac{2}{5}$ $x_2 = 6/5 = 1\frac{1}{5}$ $x_3 = 5$

$z = 205$
$x_1 = 3$ $x_2 = 3/2 = 1\frac{1}{2}$ $x_3 = 4$

$z = 584/3 = 194\frac{2}{3}$
$x_1 = 4$ $x_2 = 5/3 = 1\frac{2}{3}$ $x_3 = 4/3 = 1\frac{1}{3}$

Optimal solution:

$\widetilde{x} = (3, 3/2, 4)$

$z(\widetilde{x}) = 205$