Math 334, Solutions to selected problems

1.1(4): x_1 = acres of corn; x_2 = acres of soybeans; x_3 = acres of oats. Find $(x_1, x_2, x_3) \in \mathbb{R}^3$ that maximizes $z = 40x_1 + 30x_2 + 20x_3$ subject to the constraints:
$x_1 + x_2 + x_3 \leq 12$,
$6x_1 + 6x_2 + 2x_3 \leq 48$,
$36x_1 + 24x_2 + 18x_3 \leq 360$.

1.2(14): The slack variables are (with $x = 1$ and $y = 2$):
$u_1 = 10 - 2x - 3y = 2$,
$u_2 = 12 - 5x - y = 5$,
$u_3 = 15 - x - 5y = 4$.
Since x, y, and all slack variables are non-negative, $\mathbf{x} = (1, 2)^T$ is a feasible solution.

1.2(16): Let $r, s \in \mathbb{R}$ be real numbers satisfying $r \geq 0$, $s \geq 0$, and $r + s = 1$. Then for arbitrary real numbers $p, q \in \mathbb{R}$ we have

$$\min(p, q) \leq rp + sq \leq \max(p, q)\,.$$

For example, if $p \leq q$, then we have $\min(p, q) = p = (r+s)p \leq rp+sq \leq (r + s)q = \max(p, q)$.
Consider the half-space of $\mathbb{R}^n$ defined by the linear inequality $a_1x_1 + \cdots + a_nx_n \leq b$, or $a^Tx \leq b$ in vector notation. Assume that $x, y \in \mathbb{R}^n$ are two points contained in this half-plane, and let $r, s \in \mathbb{R}$ be as above. Set $z = rx + sy \in \mathbb{R}^n$. Then $a^Tz = a^T(rx + sy) = r(a^Tx) + s(a^Ty) \leq \max(a^Tx, a^Ty) \leq b$ shows that z belongs to the half-space. This is the essential point in the exercise.

1.3(30): Let $S_1, S_2, \ldots, S_r \subset \mathbb{R}^n$ be convex sets, and set $T = S_1 \cap S_2 \cap \cdots \cap S_r$. We must show that T is convex. Let $x, y \in T$ be two arbitrary points, and let z be a point on the line segment from x to y. We need to show that $z \in T$. Since S_i is convex, $x, y \in S_i$, and z belongs to the line segment from x to y, it follows that $z \in S_i$ for each i. This implies that $z \in S_1 \cap S_2 \cap \cdots \cap S_r = T$, as required.

1.3(32): Let $x, y \in R^n$ be points satisfying $c^Tx = c^Ty = k$. We must show that $c^Tz = k$ for any point z on the line segment from x to y. We know that any such point z can be written as $z = (1 - \lambda)x + \lambda y$ with $\lambda \in [0, 1]$. It follows that $c^Tz = c^T((1 - \lambda)x + \lambda y) = (1 - \lambda)(c^Tx) + \lambda(c^Ty) = (1 - \lambda)k + \lambda k = k$.

1.3(34): Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$. The set of solutions to $Ax \le b$ is the intersection of the m half-spaces defined by

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$

for $1 \le i \le m$. We argued above in 1.2(16) that any half-space is convex. It therefore follows from 1.3(30) that the intersection of all the half-spaces is a convex set.

(Notice that the empty set satisfies the definition of 'convex' on page 79, so there is no need to assume that the set of solutions to $Ax \le b$ is not empty.)

1.4(16): Consider a linear problem where we must maximize $z = c^T x$ on a set of feasible solutions $S \subset \mathbb{R}^n$. Assume that $x_1, x_2, \ldots, x_r \in \mathbb{R}^n$ are extreme points of S, and that z is maximal at all these points. If we let z_0 be the maximal value of z, then we have $c^T x_i = z_0$ for $1 \le i \le r$. Let $y \in \mathbb{R}^n$ be any convex combination of the points $x_1, x_2, \ldots, x_r$. We must show that $c^T y = z_0$. Since y is a convex combination, we can find non-negative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $y = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_r x_r$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$. It follows that $c^T y = c^T(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_r x_r) = \lambda_1(c^T x_1) + \lambda_2(c^T x_2) + \cdots + \lambda_r(c^T x_r) = \lambda_1 z_0 + \lambda_2 z_0 + \cdots + \lambda_r z_0 = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) z_0 = z_0$.

1.5(6): (a) Not a solution.

(b) Not a basic solution because columns 2 and 3 of the matrix are linearly dependent.

(c) Not a basic solution because too many entries are non-zero.

(d) Basic solution.

1.5(8): Canonical form:

Maximize $z = 3x + 2y$ subject to the constraints

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

and $x \ge 0$, $y \ge 0$, $u \ge 0$, $v \ge 0$.

Extreme feasible point	Basic variables	Objective function
$(4, 2, 0, 0)^T$	x, y	$z = 16$
$(3, 0, 0, 4)^T$	x, v	$z = 9$
$(0, 10, 16, 0)^T$	y, u	$z = 20$
$(0, 0, 6, 10)^T$	u, v	$z = 0$

Optimal solution: $(0, 10, 16, 0)^T$.

2.1(8): (a) Basic feasible solution: $(4, 0, 0, 0, 4, 10, 0)^T$.
(b) Entering variable: x_2
$\theta_1 = 4/(4/3) = 3$, $\theta_2 = 10/(1/3) = 30$, $\theta_3 = 4/(1/3) = 12$.
Departing variable: x_5.
The new tableau represents the basic feasible solution $(3, 3, 0, 0, 0, 9, 0)^T$.