Example

In class we did not yet discuss the easiest way to find the optimal solution to the dual problem in all cases, therefore I am including this example. Notice that you don't need to follow this procedure. You can also find the matrix B directly as discussed in class and compute B^{-1} by hand.

Primal Problem:

Maximize $z = 2x_1 - x_2 + x_3$
subject to
$$\begin{aligned} 3x_1 - 12x_2 + 3x_3 &\geq 1 \\ 2x_1 \qquad\quad - x_3 &\leq 8 \\ 2x_1 - 6x_2 + 3x_3 &= 6 \end{aligned}$$
$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

Important: All constraints must be of the form "$\geq b_i$", "$\leq b_i$", or "$= b_i$", where b_i is **non-negative**.

Dual Problem:

Minimize $z' = -w_1 + 8w_2 + 6w_3$
subject to
$-3w_1 + 2w_2 + 2w_3 \geq 2$
$12w_1 - 6w_3 \geq -1$
$-3w_1 - w_2 + 3w_3 \geq 1$
$w_1 \geq 0$, $w_2 \geq 0$, w_3 unrestricted.

Note: The first constraint of the primal problem is **negated** before we form the dual problem.

Primal problem in canonical form:

Maximize $z = 2x_1 - x_2 + x_3$
subject to
$3x_1 - 12x_2 + 3x_3 - x_4 = 1$
$2x_1 - x_3 + x_5 = 8$
$2x_1 - 6x_2 + 3x_3 = 6$
$x_j \geq 0$ for $1 \leq j \leq 5$.

Note: We have added slack variables x_4 and x_5.

Phase 1 problem:

Maximize $z'' = -y_1 - y_2$
subject to
$3x_1 - 12x_2 + 3x_3 - x_4 + y_1 = 1$
$2x_1 - x_3 + x_5 = 8$
$2x_1 - 6x_2 + 3x_3 + y_2 = 6$
$x_j \geq 0$ for $1 \leq j \leq 5$, $y_1 \geq 0$, $y_2 \geq 0$.

Note: We have added artificial variables y_1 and y_2 for the first and last constraints. For the second constraint we can use x_5 as the initial basic variable.

Phase 1:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
y_1	3	-12	3	-1	0	1	0	1
x_5	2	0	-1	0	1	0	0	8
y_2	2	-6	3	0	0	0	1	6
	0	0	0	0	0	1	1	0

Initial tableau:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
y_1	3	-12	3	-1	0	1	0	1
x_5	2	0	-1	0	1	0	0	8
y_2	2	-6	3	0	0	0	1	6
	-5	18	-6	1	0	0	0	-7

Entering variable x_3, departing variable y_1.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	1	-4	1	-1/3	0	1/3	0	1/3
x_5	3	-4	0	-1/3	1	1/3	0	25/3
y_2	-1	6	0	1	0	-1	1	5
	1	-6	0	-1	0	2	0	-5

Entering variable x_2, departing variable y_2.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	1/3	0	1	1/3	0	-1/3	2/3	11/3
x_5	7/3	0	0	1/3	1	-1/3	2/3	35/3
x_2	-1/6	1	0	1/6	0	-1/6	1/6	5/6
	0	0	0	0	0	1	1	0

Phase 2:

To make it easy to find the optimal solution to the dual problem, we keep the artificial variables during Phase 2. However, we will never use artificial variables as entering variables, and when we check the optimality of a tableau, we ignore entries of the objective row corresponding to the artificial variables.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	1/3	0	1	1/3	0	-1/3	2/3	11/3
x_5	7/3	0	0	1/3	1	-1/3	2/3	35/3
x_2	-1/6	1	0	1/6	0	-1/6	1/6	5/6
	-2	1	-1	0	0	0	0	0

Initial tableau:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	1/3	0	1	1/3	0	-1/3	2/3	11/3
x_5	7/3	0	0	1/3	1	-1/3	2/3	35/3
x_2	-1/6	1	0	1/6	0	-1/6	1/6	5/6
	-3/2	0	0	1/6	0	-1/6	1/2	17/6

Entering variables x_1, departing variables x_5.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	0	0	1	2/7	-1/7	-2/7	4/7	2
x_1	1	0	0	1/7	3/7	-1/7	2/7	5
x_2	0	1	0	4/21	1/14	-4/21	3/14	5/3
	0	0	0	8/21	9/14	-8/21	13/14	31/3

This tableau represents the optimal solution since we ignore the signs of the entries in the objective row corresponding to artificial variables.

Optimal solutions:

Optimal solution to the primary problem:

$$(x_1, x_2, x_3) = (5, 5/3, 2)\,.$$

Matrix of columns in initial tableau (from Phase 1) corresponding to final basic variables (x_3, x_1, x_2):

$$B = \begin{bmatrix} 3 & 3 & -12 \\ -1 & 2 & 0 \\ 3 & 2 & -6 \end{bmatrix}$$

We get the inverse of B by reading columns of final tableau corresponding to initial basic variables (y_1, x_5, y_2):

$$B^{-1} = \begin{bmatrix} -2/7 & -1/7 & 4/7 \\ -1/7 & 3/7 & 2/7 \\ -4/21 & 1/14 & 3/14 \end{bmatrix}$$

Vector of entries of c corresponding to the final basic variables (x_3, x_1, x_2):

$$c_B^T = (1, 2, -1)$$

Optimal solution to the dual of the problem in canonical form:

$$\widehat{w}^T = c_B^T B^{-1} = (-8/21, 9/14, 13/14)$$

This vector can also be obtained as follows: Add $c^T = (2, -1, 1, 0, 0, 0, 0)$ to the final objective row of phase 2, this gives

$$(2, -1, 1, 8/21, 9/14, -8/21, 13/14)\,,$$

then extract the coordinates corresponding to the initial basic variables (y_1, x_5, y_2). We will discuss why in class on Tuesday! You are encouraged to use both methods and check that they give the same result.

Note: $\widehat{w}^T$ is NOT the optimal solution to the dual of the primal problem!

In fact, since dualizing the primal problem involved negating the first constraint, thereby negating b_1, we need to negate $\widehat{w}_1$ to obtain the optimal solution of the dual of the primal problem.

Optimal solution to the dual problem:

$$w^T = (8/21, 9/14, 13/14)$$

To check that we got everything right, we can verify that $x^T = (2, 5, 5/3)$ is a feasible solution to the primal problem, that $w^T = (8/21, 9/14, 13/14)$ is a solution to the dual problem, and that $z(x) = 31/3 = z'(w)$.