Linear problem P:

Maximize $z = -5x_1 + 2x_2 - 5x_3 - x_4 - x_5$
subject to

$$\begin{array}{rcl} -3x_1 + x_2 + x_3 \phantom{{}+ 2x_4 + 3x_5} & = & 3 \\ -3x_1 + x_2 + 12x_3 + x_4 + 2x_5 & = & 5 \\ x_1 \phantom{{}+ x_2} + 18x_3 + 2x_4 + 3x_5 & = & 4 \end{array}$$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Challenge: Not clear how to find a basic feasible solution (BFS).

Suppose we know that one BFS uses basic variables x_1, x_2, x_3.

How can we find a tableau representing it?

$$\begin{array}{|ccccc|c|}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
-3 & 1 & 1 & 0 & 0 & 3 \\
-3 & 1 & 12 & 1 & 2 & 5 \\
1 & 0 & 18 & 2 & 3 & 4 \\
\hline
5 & -2 & 5 & 1 & 1 & 0 \\
\hline
\end{array}$$

$$\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
\enclose{circle}{-3} & 1 & 1 & 0 & 0 & 3 \\
-3 & \enclose{circle}{1} & 12 & 1 & 2 & 5 \\
1 & 0 & \enclose{circle}{18} & 2 & 3 & 4 \\
\hline
5 & -2 & 5 & 1 & 1 & 0 \\
\hline
\end{array}$$

$$\begin{array}{c|ccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_1 & \textcircled{1} & 0 & 0 & 4/11 & -3/11 & 8/11 \\
x_2 & 0 & \textcircled{1} & 0 & 1 & -1 & 5 \\
x_3 & 0 & 0 & \textcircled{1} & 1/11 & 2/11 & 2/11 \\
\hline
 & 0 & 0 & 0 & 8/11 & -6/11 & 60/11 \\
\hline
\end{array}$$

What if we randomly choose three basic variables?

x_1	x_2	x_3	x_4	x_5	
-3	1	1	0	0	3
-3	1	12	1	2	5
1	0	18	2	3	4
5	-2	5	1	1	0

$$
\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
-3 & \boxed{1} & 1 & 0 & 0 & 3 \\
-3 & 1 & \boxed{12} & 1 & 2 & 5 \\
1 & 0 & 18 & 2 & \boxed{3} & 4 \\
\hline
5 & -2 & 5 & 1 & 1 & 0
\end{array}
$$

	x_1	x_2	x_3	x_4	x_5	
x_2	-11/3	1	0	-1/3	0	7/3
x_3	2/3	0	1	1/3	0	2/3
x_5	-11/3	0	0	-4/3	1	-8/3
	-2	0	0	0	0	4

	x_1	x_2	x_3	x_4	x_5	
x_2	-11/3	1	0	-1/3	0	7/3
x_3	2/3	0	1	1/3	0	2/3
x_5	-11/3	0	0	-4/3	1	-8/3
	-2	0	0	0	0	4

$$\begin{array}{c|ccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_4 & 11 & -3 & 0 & \text{①} & 0 & -7 \\
x_3 & -3 & 1 & \text{①} & 0 & 0 & 3 \\
x_5 & 11 & -4 & 0 & 0 & \text{①} & -12 \\
\hline
 & -2 & 0 & 0 & 0 & 0 & 4 \\
\hline
\end{array}$$

Imagine a random linear problem with 50 variables and 25 constraints.

Number of ways to choose 25 basic variables: $\binom{50}{25} = 126410606437752$

Probability of guessing basic variables of some BFS: $\left(\frac{1}{2}\right)^{25} = \frac{1}{33554432}$

Linear problem P:

Maximize $z = -5x_1 + 2x_2 - 5x_3 - x_4 - x_5$
subject to

$$\begin{array}{rcl} -3x_1 + x_2 + x_3 \phantom{{}+ x_4 + 2x_5} & = & 3 \\ -3x_1 + x_2 + 12x_3 + x_4 + 2x_5 & = & 5 \\ x_1 \phantom{{}+ x_2} + 18x_3 + 2x_4 + 3x_5 & = & 4 \end{array}$$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Challenge: Not clear how to find BFS.

Solution: Use phase 1 in two-phase algorithm!

Set of feasible solutions:

$$S = \left\{ x \in \mathbb{R}^5 \;\middle|\; \begin{array}{rcl} -3x_1 + x_2 + x_3 \phantom{{}+2x_4+3x_5} & = & 3 \\ -3x_1 + x_2 + 12x_3 + x_4 + 2x_5 & = & 5 \\ x_1 \phantom{{}+x_2} + 18x_3 + 2x_4 + 3x_5 & = & 4 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 & & \end{array} \right\}$$

Consider larger set of points:

$$S' = \left\{ x \in \mathbb{R}^5 \;\middle|\; \begin{array}{rcl} -3x_1 + x_2 + \quad x_3 & \leq & 3 \\ -3x_1 + x_2 + 12x_3 + \quad x_4 + 2x_5 & \leq & 5 \\ x_1 \qquad + 18x_3 + 2x_4 + 3x_5 & \leq & 4 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 & & \end{array} \right\}$$

Note: $0 \in S'$

Define
$$\begin{aligned} y_1 &= 3 - (-3x_1 + x_2 + x_3) \\ y_2 &= 5 - (-3x_1 + x_2 + 12x_3 + x_4 + 2x_5) \\ y_3 &= 4 - (x_1 + 18x_3 + 2x_4 + 3x_5) \end{aligned}$$

Minimize $z' = y_1 + y_2 + y_3$

Phase 1 problem:

Maximize $-z' = -y_1 - y_2 - y_3$
subject to

$$\begin{array}{rcl} -3x_1 + x_2 + \quad x_3 \qquad\qquad\qquad + y_1 \qquad\qquad & = & 3 \\ -3x_1 + x_2 + 12x_3 + \ x_4 + 2x_5 \qquad + y_2 \qquad & = & 5 \\ x_1 \qquad + 18x_3 + 2x_4 + 3x_5 \qquad\qquad + y_3 & = & 4 \end{array}$$

$x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \geq 0$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	-3	1	1	0	0	1	0	0	3
y_2	-3	1	12	1	2	0	1	0	5
y_3	1	0	18	2	3	0	0	1	4
	0	0	0	0	0	1	1	1	0

$$
\begin{array}{c|cccccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & \textcircled{1} & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & \textcircled{1} & 0 & 5 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & \textcircled{1} & 4 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
\end{array}
$$

$$
\begin{array}{c|cccccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & \textcircled{1} & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & \textcircled{1} & 0 & 5 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & \textcircled{1} & 4 \\
\hline
 & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\
\hline
\end{array}
$$

$$\begin{array}{c|cccccccc|c|}
 & & \downarrow & & & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\ \cline{2-10}
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & 1 & 0 & 5 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\ \cline{2-10}
 & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\ \cline{2-10}
\end{array}$$

$$
\begin{array}{cc|cccccccc|c|}
 & & & \downarrow & & & & & & & \\
 & & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\cline{3-11}
\leftarrow & y_1 & -3 & \textcircled{1} & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
 & y_2 & -3 & 1 & 12 & 1 & 2 & 0 & 1 & 0 & 5 \\
 & y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\
\cline{3-11}
 & & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\
\cline{3-11}
\end{array}
$$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_2	-3	1	1	0	0	1	0	0	3
y_2	0	0	11	1	2	-1	1	0	2
y_3	1	0	18	2	3	0	0	1	4
	-1	0	-29	-3	-5	2	0	0	-6

$$\begin{array}{c|cccccccc|c|}
 & \downarrow & & & & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
x_2 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & 0 & 0 & 11 & 1 & 2 & -1 & 1 & 0 & 2 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\
\hline
 & -1 & 0 & -29 & -3 & -5 & 2 & 0 & 0 & -6 \\
\hline
\end{array}$$

$$\begin{array}{rccccccccc}
 & \downarrow & & & & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\cline{2-10}
\multicolumn{1}{r|}{x_2} & -3 & 1 & 1 & 0 & 0 & 1 & 0 & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{3} \\
\multicolumn{1}{r|}{y_2} & 0 & 0 & 11 & 1 & 2 & -1 & 1 & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{2} \\
\multicolumn{1}{r|}{\leftarrow y_3} & \textcircled{1} & 0 & 18 & 2 & 3 & 0 & 0 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} \\
\cline{2-10}
\multicolumn{1}{r|}{} & -1 & 0 & -29 & -3 & -5 & 2 & 0 & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{-6} \\
\cline{2-10}
\end{array}$$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_2	0	1	55	6	9	1	0	3	15
y_2	0	0	11	1	2	-1	1	0	2
x_1	①	0	18	2	3	0	0	1	4
	0	0	-11	-1	-2	2	0	1	-2

			↓						
	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_2	0	1	55	6	9	1	0	3	15
y_2	0	0	11	1	2	-1	1	0	2
x_1	1	0	18	2	3	0	0	1	4
	0	0	-11	-1	-2	2	0	1	-2

$$
\begin{array}{rc|cccccccc|c}
 & & & & \downarrow & & & & & & \\
 & & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
 & x_2 & 0 & 1 & 55 & 6 & 9 & 1 & 0 & 3 & 15 \\
\leftarrow & y_2 & 0 & 0 & \textcircled{11} & 1 & 2 & -1 & 1 & 0 & 2 \\
 & x_1 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\
\hline
 & & 0 & 0 & -11 & -1 & -2 & 2 & 0 & 1 & -2 \\
\hline
\end{array}
$$

$$\begin{array}{c|cccccccc|c|}
 & & & \downarrow & & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
x_2 & 0 & 1 & 55 & 6 & 9 & 1 & 0 & 3 & 15 \\
\leftarrow & 0 & 0 & \textcircled{1} & 1/11 & 2/11 & -1/11 & 1/11 & 0 & 2/11 \\
x_1 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\
\hline
 & 0 & 0 & -11 & -1 & -2 & 2 & 0 & 1 & -2 \\
\hline
\end{array}$$

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
x_2 & 0 & 1 & 0 & 1 & -1 & 6 & -5 & 3 & 5 \\
x_3 & 0 & 0 & \textcircled{1} & 1/11 & 2/11 & -1/11 & 1/11 & 0 & 2/11 \\
x_1 & 1 & 0 & 0 & 4/11 & -3/11 & 18/11 & -18/11 & 1 & 8/11 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
\end{array}$$

The following tableau encodes the set S' (with extra slack coordinates).

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	-3	1	1	0	0	1	0	0	3
y_2	-3	1	12	1	2	0	1	0	5
y_3	1	0	18	2	3	0	0	1	4

Without columns of y_1, y_2, y_3, it encodes the set of feasible solutions:

$$S = \left\{ x \in \mathbb{R}^5 \;\middle|\; \begin{array}{rcl} -3x_1 + x_2 + x_3 \phantom{{}+2x_4+3x_5} & = & 3 \\ -3x_1 + x_2 + 12x_3 + x_4 + 2x_5 & = & 5 \\ x_1 \phantom{{}+x_2} + 18x_3 + 2x_4 + 3x_5 & = & 4 \\ \multicolumn{3}{c}{x_1, x_2, x_3, x_4, x_5 \geq 0} \end{array} \right\}$$

The following tableau encodes the set S' (with extra slack coordinates).

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_2	0	1	0	1	-1	6	-5	3	5
x_3	0	0	1	1/11	2/11	-1/11	1/11	0	2/11
x_1	1	0	0	4/11	-3/11	18/11	-18/11	1	8/11

Without columns of y_1, y_2, y_3, it encodes the set of feasible solutions:

$$S = \left\{ x \in \mathbb{R}^5 \;\middle|\; \begin{array}{rcl} -3x_1 + x_2 + \quad x_3 & = & 3 \\ -3x_1 + x_2 + 12x_3 + \;\; x_4 + 2x_5 & = & 5 \\ x_1 \qquad + 18x_3 + 2x_4 + 3x_5 & = & 4 \\ \multicolumn{3}{c}{x_1, x_2, x_3, x_4, x_5 \geq 0} \end{array} \right\}$$

Original linear problem P is encoded by the tableau:

$$\begin{array}{c|ccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_2 & 0 & 1 & 0 & 1 & -1 & 5 \\
x_3 & 0 & 0 & 1 & 1/11 & 2/11 & 2/11 \\
x_1 & 1 & 0 & 0 & 4/11 & -3/11 & 8/11 \\
\hline
 & 5 & -2 & 5 & 1 & 1 & 0 \\
\hline
\end{array}$$

Current BFS: $x = (\frac{8}{11}, 5, \frac{2}{11}, 0, 0)^T$

Note: We use the original objective function:

$$z = -5x_1 + 2x_2 - 5x_3 - x_4 - x_5$$

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	0	1	-1	5
x_3	0	0	1	1/11	2/11	2/11
x_1	1	0	0	4/11	-3/11	8/11
	5	-2	5	1	1	0

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	0	1	-1	5
x_3	0	0	1	1/11	2/11	2/11
x_1	1	0	0	4/11	-3/11	8/11
	5	-2	5	1	1	0

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	0	1	-1	5
x_3	0	0	1	1/11	2/11	2/11
x_1	1	0	0	4/11	-3/11	8/11
	0	0	0	8/11	-6/11	60/11

$$\begin{array}{c|ccccc|c|}
 & & & & & \downarrow & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_2 & 0 & 1 & 0 & 1 & -1 & 5 \\
x_3 & 0 & 0 & 1 & 1/11 & 2/11 & 2/11 \\
x_1 & 1 & 0 & 0 & 4/11 & -3/11 & 8/11 \\
\hline
 & 0 & 0 & 0 & 8/11 & -6/11 & 60/11 \\
\hline
\end{array}$$

$$
\begin{array}{cc|ccccc|c}
 & & & & & & \downarrow & \\
 & & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
 & x_2 & 0 & 1 & 0 & 1 & -1 & 5 \\
\leftarrow & x_3 & 0 & 0 & 1 & 1/11 & \enclose{circle}{2/11} & 2/11 \\
 & x_1 & 1 & 0 & 0 & 4/11 & -3/11 & 8/11 \\
\hline
 & & 0 & 0 & 0 & 8/11 & -6/11 & 60/11 \\
\hline
\end{array}
$$

$$\begin{array}{c|ccccc|c|}
 & & & & & \downarrow & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_2 & 0 & 1 & 0 & 1 & -1 & 5 \\
\leftarrow & 0 & 0 & 11/2 & 1/2 & \boxed{1} & 1 \\
x_1 & 1 & 0 & 0 & 4/11 & -3/11 & 8/11 \\
\hline
 & 0 & 0 & 0 & 8/11 & -6/11 & 60/11 \\
\hline
\end{array}$$

$$\begin{array}{c|ccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_2 & 0 & 1 & 11/2 & 3/2 & 0 & 6 \\
x_5 & 0 & 0 & 11/2 & 1/2 & \enclose{circle}{1} & 1 \\
x_1 & 1 & 0 & 3/2 & 1/2 & 0 & 1 \\
\hline
 & 0 & 0 & 3 & 1 & 0 & 6 \\
\hline
\end{array}$$

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	11/2	3/2	0	6
x_5	0	0	11/2	1/2	1	1
x_1	1	0	3/2	1/2	0	1
	0	0	3	1	0	6

Optimal solution: $x = (1, 6, 0, 0, 1)^T \quad ; \quad z(x) = 6$

Phase 1 problem:

Maximize $-z' = -y_1 - y_2 - y_3$
subject to

$$\begin{array}{rcl} -3x_1 + x_2 + x_3 + y_1 & = & 3 \\ -3x_1 + x_2 + 12x_3 + x_4 + 2x_5 + y_2 & = & 5 \\ x_1 + 18x_3 + 2x_4 + 3x_5 + y_3 & = & 4 \end{array}$$

$x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \geq 0$

Let's try it again!

$$
\begin{array}{c|cccccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\cline{2-10}
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & 1 & 0 & 5 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\
\cline{2-10}
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\cline{2-10}
\end{array}
$$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	-3	1	1	0	0	①	0	0	3
y_2	-3	1	12	1	2	0	①	0	5
y_3	1	0	18	2	3	0	0	①	4
	0	0	0	0	0	1	1	1	0

$$
\begin{array}{c|cccccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & \textcircled{1} & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & \textcircled{1} & 0 & 5 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & \textcircled{1} & 4 \\
\hline
 & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\
\hline
\end{array}
$$

$$\begin{array}{c|cccccccc|c|}
\multicolumn{1}{c}{} & & & & \downarrow & & & & & \multicolumn{1}{c}{} \\
\multicolumn{1}{c}{} & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \multicolumn{1}{c}{} \\
\cline{2-10}
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & 1 & 0 & 5 \\
y_3 & 1 & 0 & 18 & 2 & 3 & 0 & 0 & 1 & 4 \\
\cline{2-10}
 & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\
\cline{2-10}
\end{array}$$

$$\begin{array}{r|cccccccc|c|}
 & & & & \downarrow & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & 1 & 0 & 5 \\
\leftarrow y_3 & 1 & 0 & 18 & \textcircled{2} & 3 & 0 & 0 & 1 & 4 \\
\hline
 & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\
\hline
\end{array}$$

$$\begin{array}{c|cccccccc|c}
 & & & & \downarrow & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -3 & 1 & 12 & 1 & 2 & 0 & 1 & 0 & 5 \\
\leftarrow & 1/2 & 0 & 9 & \boxed{1} & 3/2 & 0 & 0 & 1/2 & 2 \\
\hline
 & 5 & -2 & -31 & -3 & -5 & 0 & 0 & 0 & -12 \\
\hline
\end{array}$$

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -7/2 & 1 & 3 & 0 & 1/2 & 0 & 1 & -1/2 & 3 \\
x_4 & 1/2 & 0 & 9 & \text{\textcircled{1}} & 3/2 & 0 & 0 & 1/2 & 2 \\
\hline
 & 13/2 & -2 & -4 & 0 & -1/2 & 0 & 0 & 3/2 & -6 \\
\hline
\end{array}$$

$$\begin{array}{c|cccccccc|c|}
 & & \downarrow & & & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
y_2 & -7/2 & 1 & 3 & 0 & 1/2 & 0 & 1 & -1/2 & 3 \\
x_4 & 1/2 & 0 & 9 & 1 & 3/2 & 0 & 0 & 1/2 & 2 \\
\hline
 & 13/2 & -2 & -4 & 0 & -1/2 & 0 & 0 & 3/2 & -6 \\
\hline
\end{array}$$

$$\begin{array}{cc|cccccccc|c|}
 & & & \downarrow & & & & & & & \\
 & & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\cline{3-11}
 & y_1 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\
\leftarrow & y_2 & -7/2 & \textcircled{1} & 3 & 0 & 1/2 & 0 & 1 & -1/2 & 3 \\
 & x_4 & 1/2 & 0 & 9 & 1 & 3/2 & 0 & 0 & 1/2 & 2 \\
\cline{3-11}
 & & 13/2 & -2 & -4 & 0 & -1/2 & 0 & 0 & 3/2 & -6 \\
\cline{3-11}
\end{array}$$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	1	-1	1/2	0
x_2	-7/2	①	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	1	3/2	0	0	1/2	2
	-1/2	0	2	0	1/2	0	2	1/2	0

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	1	-1	1/2	0
x_2	-7/2	1	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	1	3/2	0	0	1/2	2
	-1/2	0	2	0	1/2	0	2	1/2	0

Optimal solution to Phase 1 problem:

$(x, y) = (0, 3, 0, 2, 0, 0, 0, 0) \quad ; \quad z'(x, y) = 0$

Note: The artificial variable y_1 is still a basic variable.

Linear problem P is represented by:

$$\begin{array}{c|ccccc|c|} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ \cline{2-7} & 1/2 & 0 & -2 & 0 & -1/2 & 0 \\ x_2 & -7/2 & 1 & 3 & 0 & 1/2 & 3 \\ x_4 & 1/2 & 0 & 9 & 1 & 3/2 & 2 \\ \cline{2-7} & 5 & -2 & 5 & 1 & 1 & 0 \\ \cline{2-7} \end{array}$$

Pivot an arbitrary non-zero entry in first row to add missing basic variable!

	x_1	x_2	x_3	x_4	x_5	
	1/2	0	-2	0	-1/2	0
x_2	-7/2	1	3	0	1/2	3
x_4	1/2	0	9	1	3/2	2
	5	-2	5	1	1	0

	x_1	x_2	x_3	x_4	x_5	
x_5	-1	0	4	0	①	0
x_2	-3	①	1	0	0	3
x_4	2	0	3	①	0	2
	-2	0	0	0	0	4

$$\begin{array}{c|ccccc|c}
 & \downarrow & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_5 & -1 & 0 & 4 & 0 & 1 & 0 \\
x_2 & -3 & 1 & 1 & 0 & 0 & 3 \\
x_4 & 2 & 0 & 3 & 1 & 0 & 2 \\
\hline
 & -2 & 0 & 0 & 0 & 0 & 4
\end{array}$$

$$
\begin{array}{c|ccccc|c|}
 & \downarrow & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_5 & -1 & 0 & 4 & 0 & 1 & 0 \\
x_2 & -3 & 1 & 1 & 0 & 0 & 3 \\
\leftarrow x_4 & \textcircled{2} & 0 & 3 & 1 & 0 & 2 \\
\hline
 & -2 & 0 & 0 & 0 & 0 & 4 \\
\hline
\end{array}
$$

	↓ x_1	x_2	x_3	x_4	x_5	
x_5	-1	0	4	0	1	0
x_2	-3	1	1	0	0	3
←	①	0	3/2	1/2	0	1
	-2	0	0	0	0	4

$$\begin{array}{c|ccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\cline{2-7}
x_5 & 0 & 0 & 11/2 & 1/2 & 1 & 1 \\
x_2 & 0 & 1 & 11/2 & 3/2 & 0 & 6 \\
x_1 & \enclose{circle}{1} & 0 & 3/2 & 1/2 & 0 & 1 \\
\cline{2-7}
 & 0 & 0 & 3 & 1 & 0 & 6 \\
\cline{2-7}
\end{array}$$

	x_1	x_2	x_3	x_4	x_5	
x_5	0	0	11/2	1/2	1	1
x_2	0	1	11/2	3/2	0	6
x_1	1	0	3/2	1/2	0	1
	0	0	3	1	0	6

Optimal solution: $x = (1, 6, 0, 0, 1)^T \quad ; \quad z(x) = 6$

Back to final tableau from Phase 1:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	1	-1	1/2	0
x_2	-7/2	1	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	1	3/2	0	0	1/2	2
	-1/2	0	2	0	1/2	0	2	1/2	0

Optimal solution to Phase 1 problem:

$$(x, y) = (0, 3, 0, 2, 0, 0, 0, 0) \quad ; \quad z'(x, y) = 0$$

Note: The artificial variable y_1 is still a basic variable.

Alternative way to do phase 2:

Maximize objective function $z = -5x_1 + 2x_2 - 5x_3 - x_4 - x_5$ (from original problem P) on set S' of feasible solutions to Phase 1 problem.

Never choose an artificial vairable as entering variable!

(Ignore negative entries in objective row under artificial variables.)

Always choose artificial variable as departing variable if possible!

This way we only work with feasible solutions to original problem P.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	1	-1	1/2	0
x_2	-7/2	1	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	1	3/2	0	0	1/2	2
	-1/2	0	2	0	1/2	0	2	1/2	0

$$\begin{array}{c|cccccccc|c|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
y_1 & 1/2 & 0 & -2 & 0 & -1/2 & 1 & -1 & 1/2 & 0 \\
x_2 & -7/2 & 1 & 3 & 0 & 1/2 & 0 & 1 & -1/2 & 3 \\
x_4 & 1/2 & 0 & 9 & 1 & 3/2 & 0 & 0 & 1/2 & 2 \\
\hline
 & 5 & -2 & 5 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

Use original objective function $z = -5x_1 + 2x_2 - 5x_3 - x_4 - x_5.$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	①	-1	1/2	0
x_2	-7/2	①	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	①	3/2	0	0	1/2	2
	5	-2	5	1	1	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	$\textcircled{1}$	-1	1/2	0
x_2	-7/2	$\textcircled{1}$	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	$\textcircled{1}$	3/2	0	0	1/2	2
	-5/2	0	2	0	1/2	0	2	-3/2	4

	↓								
	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	1/2	0	-2	0	-1/2	1	-1	1/2	0
x_2	-7/2	1	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	1	3/2	0	0	1/2	2
	-5/2	0	2	0	1/2	0	2	-3/2	4

$$\begin{array}{c|cccccccc|c}
 & \downarrow & & & & & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
\leftarrow y_1 & \boxed{1/2} & 0 & -2 & 0 & -1/2 & 1 & -1 & 1/2 & 0 \\
x_2 & -7/2 & 1 & 3 & 0 & 1/2 & 0 & 1 & -1/2 & 3 \\
x_4 & 1/2 & 0 & 9 & 1 & 3/2 & 0 & 0 & 1/2 & 2 \\
\hline
 & -5/2 & 0 & 2 & 0 & 1/2 & 0 & 2 & -3/2 & 4 \\
\hline
\end{array}$$

	↓ x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
←	①	0	-4	0	-1	2	-2	1	0
x_2	-7/2	1	3	0	1/2	0	1	-1/2	3
x_4	1/2	0	9	1	3/2	0	0	1/2	2
	-5/2	0	2	0	1/2	0	2	-3/2	4

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_1	1	0	-4	0	-1	2	-2	1	0
x_2	0	1	-11	0	-3	7	-6	3	3
x_4	0	0	11	1	2	-1	1	0	2
	0	0	-8	0	-2	5	-3	1	4

$$\begin{array}{cccccccccc}
 & & & & & \downarrow & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\cline{2-10}
x_1 & \multicolumn{1}{|c}{1} & 0 & -4 & 0 & -1 & 2 & -2 & 1 & \multicolumn{1}{|c|}{0} \\
x_2 & \multicolumn{1}{|c}{0} & 1 & -11 & 0 & -3 & 7 & -6 & 3 & \multicolumn{1}{|c|}{3} \\
x_4 & \multicolumn{1}{|c}{0} & 0 & 11 & 1 & 2 & -1 & 1 & 0 & \multicolumn{1}{|c|}{2} \\
\cline{2-10}
 & \multicolumn{1}{|c}{0} & 0 & -8 & 0 & -2 & 5 & -3 & 1 & \multicolumn{1}{|c|}{4} \\
\cline{2-10}
\end{array}$$

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & \overset{\downarrow}{x_5} & y_1 & y_2 & y_3 & \\
\hline
x_1 & 1 & 0 & -4 & 0 & -1 & 2 & -2 & 1 & 0 \\
x_2 & 0 & 1 & -11 & 0 & -3 & 7 & -6 & 3 & 3 \\
\leftarrow x_4 & 0 & 0 & 11 & 1 & \textcircled{2} & -1 & 1 & 0 & 2 \\
\hline
 & 0 & 0 & -8 & 0 & -2 & 5 & -3 & 1 & 4 \\
\hline
\end{array}$$

$$\begin{array}{c|cccccccc|c|}
 & & & & & \downarrow & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \\
\hline
x_1 & 1 & 0 & -4 & 0 & -1 & 2 & -2 & 1 & 0 \\
x_2 & 0 & 1 & -11 & 0 & -3 & 7 & -6 & 3 & 3 \\
\leftarrow & 0 & 0 & 11/2 & 1/2 & \enclose{circle}{1} & -1/2 & 1/2 & 0 & 1 \\
\hline
 & 0 & 0 & -8 & 0 & -2 & 5 & -3 & 1 & 4 \\
\hline
\end{array}$$

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_1	1	0	3/2	1/2	0	3/2	-3/2	1	1
x_2	0	1	11/2	3/2	0	11/2	-9/2	3	6
x_5	0	0	11/2	1/2	①	-1/2	1/2	0	1
	0	0	3	1	0	4	-2	1	6

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_1	1	0	3/2	1/2	0	3/2	-3/2	1	1
x_2	0	1	11/2	3/2	0	11/2	-9/2	3	6
x_5	0	0	11/2	1/2	1	-1/2	1/2	0	1
	0	0	3	1	0	4	-2	1	6

Optimal solution: $(x, y) = (1, 6, 0, 0, 1, 0, 0, 0)^T \quad ; \quad z(x, y) = 6$

Usual optimal solution: $x = (1, 6, 0, 0, 1) \quad ; \quad z(x) = 6$