Linear Optimization   First class   9/1/2015

12-1:20 PM,   ARC-107

Anders Buch, Hill 234, asbuch@math.rutgers.edu

Course web: math.rutgers.edu/~asbuch/linopt-f15/

office hours: TBA

~~Midterm dates: TBA~~

Grading Policy:

Midterm 1 :	22 %	Tue 10/6 in class
Midterm 2 :	22 %	Tue 11/10 in class
Homework :	12 %	
Final :	44 %	

Linear Programming:

Basic problem: Optimize (find max/min) linear function in several variables that must sat. linear (in) equalities.

Example: Hybrid car. Specs:

	Gas	Electric
Price of driving 1 hour:	$5	$2
Speed	65 mph	50 mph

Constraints: For each hour driven on gas, electricity will be generated for driving 2 hours on elect.

Q: How far can you drive if you have $20?

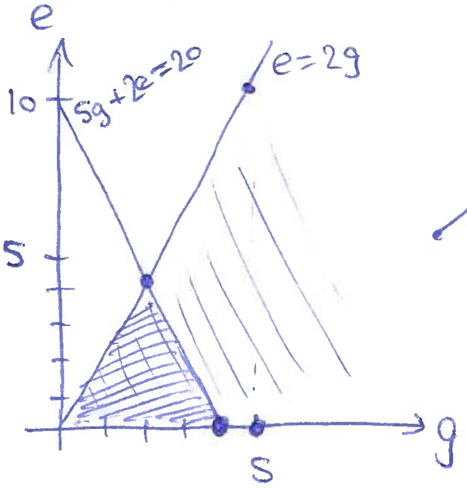
g = hours driven on gas

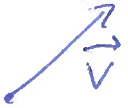
e = hours driven on electricity.

$5g + 2e \leq 20$ $\quad g \geq 0, \ e \geq 0$

$e \leq 2g$

Maximize distance $d = 65g + 50e$





Maximize $\vec{v} = (65, 50)$

Optimal point:

$5g + 2e = 20$

$e = 2g$

$g = \frac{20}{9}, \quad e = \frac{40}{9}$

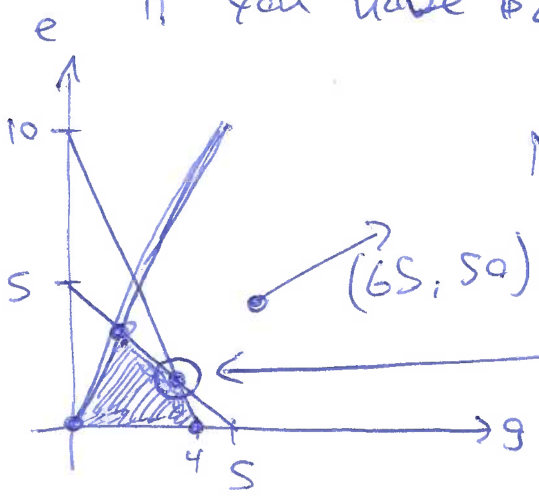
$d = 65 \cdot \frac{20}{9} + 50 \cdot \frac{40}{9} = \frac{1100}{3}$

≈ 367 miles.

Q2: How far can you drive in 5 hours if you have \$20 ?

$\boxed{\text{Extra ieq: } g + e \leq 5}$

Maximize $\vec{v} = (65, 50)$.



optimal point

Note: Different optimal point if elec. speed > gas speed.

General linear problem:

Find values of $x_1, \dots, x_n$ that will maximize (or minimize) linear function $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to linear restrictions

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \quad \le \text{ or } \ge \text{ or } = \quad b_1$$
$$a_{21}x_1 + \quad\quad + a_{2n}x_n \quad \le \quad \ge \quad = \quad b_2$$
$$\vdots \quad\quad \vdots \quad\quad \vdots$$
$$a_{m1}x_1 + \quad\quad + a_{mn}x_n \quad \le \quad \ge \quad = \quad b_m$$

Example: Maximize $z = 65x_1 + 50x_2$ subject to

$x_1 \ge 0$

$x_2 \ge 0$

$5x_1 + 2x_2 \le 20$

$2x_1 - x_2 \ge 0$

$x_1 + x_2 = 5$

Note: General problem does not require $x_i \ge 0$ for each i.

~~Linear Problem in Standard Form:~~

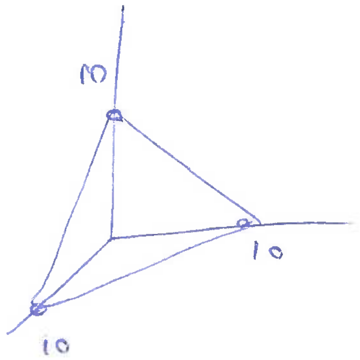
~~Find values of~~

Example ~~Maximize~~

Find values of x, y, z that will minimize $x + 2y + 3z$ subj. to

$x + y + z = 10$

$x + y \le 10 \quad y + z \le 11 \quad z + x \le 12$



Note: x, y, z may be NEGATIVE!

Linear Problem in Standard Form:

Find values of $x_1, x_2, \to x_n$ that will

maximize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_j \geq 0 \text{ for } 1 \leq j \leq n.$$

Claim Every linear problem can be solved by solving a corresp. problem in standard form.

min/max: Minimize $z = c_1x_1 + \dots + c_nx_n$.

Same as maximize

$$z' = -c_1x_1 - c_2x_2 - \dots - c_nx_n.$$

$\geq$ to $\leq$: $a_1x_1 + \dots + a_nx_n \geq b$

same as $-a_1x_1 - \dots - a_nx_n \leq -b$

$=$ to $\leq$: $a_1x_1 + \dots + a_nx_n = b_n$

same as $a_1x_1 + \dots + a_nx_n \leq b_n$

AND

$-a_1x_1 - \dots - a_nx_n \leq -b_n$.

Unconstrained variables :

Suppose that $x_j \geq 0$ is NOT a requirement.

REPLACE x_j with TWO NEW variables x_j^+, x_j^-.

Such that $x_j = x_j^+ - x_j^-$, $x_j^+ \geq 0$, $x_j^- \geq 0$.

Example Max. $x + 2y + 3z$ subj. to

$$x + y + z = 10, \quad x + y \leq 10, \quad y + z \leq 11, \quad z + x \leq 12.$$

Convert to Std Form :
$$x = x^+ - x^-$$
$$y = y^+ - y^-$$
$$z = z^+ - z^-.$$

Maximize $x^+ - x^- + 2y^+ - 2y^- + 3z^+ - 3z^-$

Subject to constraints

$$x^+ - x^- + y^+ - y^- + z^+ - z^- \leq 10$$
$$x^+ - x^- + y^+ - y^- \leq 10$$
$$y^+ - y^- + z^+ - z^- \leq 11$$
$$z^+ - z^- + x^+ - x^- \leq 12$$

$$x^- - x^+ + y^- - y^+ + z^- - z^+ \leq -10.$$

$$x^+, x^-, y^+, y^-, z^+, z^- \geq 0.$$

Example Maximize x subject to constraints

$$-1 \le x \le 1.$$

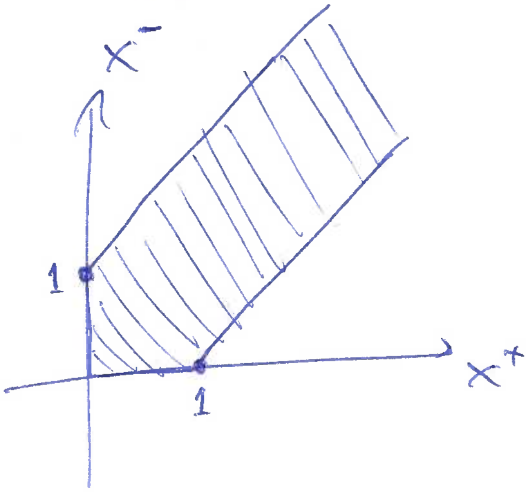
Convert to std. form: $x = x^+ - x^-$.

Maximize $x^+ - x^-$ subj. to constr.

$$x^+ - x^- \le 1$$

$$x^- - x^+ \le 1$$

$$x^+ \ge 0, \quad x^- \ge 0.$$



Note We pay the price that the bounded interval $[-1, 1]$, $-1 \le x \le 1$ is replaced with unbounded figure.

Linear Problem in Canonical Form

Find values of $x_1, \ldots, x_n$ that will

maximize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subj. to constraints

$$\begin{array}{ccc} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_n \end{array}$$

$$x_j \ge 0 \quad \text{for} \quad 1 \le j \le n.$$

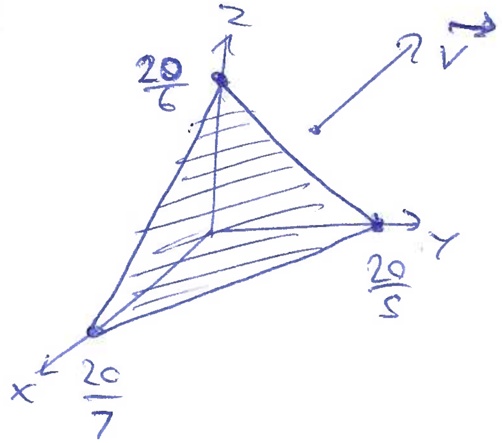
Example Maximize $3x + 4y + 5z$ subj. to constraints

$$7x + 5y + 6z = 20$$

$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

Maximize

$\vec{v} = (3, 4, 5)$



Claim Every linear problem can be solved by solving a problem in canonical form.

May assume problem in Std. form:

$$\text{Max} \quad c_1x_1 + \cdots + c_nx_n \quad \text{subj. to constr.}$$

$$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$$

$$x_j \geq 0, \quad 1 \leq j \leq n.$$

Introduce variables $y_i = b_i - (a_{i1}x_1 + \cdots + a_{in}x_n)$ for $1 \leq i \leq m$.

~~Max~~ Find $x_1, \ldots, x_n, y_1, \ldots, y_m$ that max $c_1x_1 + \cdots + c_nx_n$ subj. to

$$a_{i1}x_1 + \cdots + a_{in}x_n + y_i = b_i \quad \text{for } 1 \leq i \leq m$$

$$x_1, \ldots, x_n, y_1, \ldots, y_m \geq 0.$$

Linear Problem is Standard Form

Find values of $x_1, \to x_n$ that maximizes $\underbrace{z = c_1x_1 + \cdots + c_nx_n}_{\text{objective function}}$
subject to constraints

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\le b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\le b_m \\ x_j \ge 0 \quad \text{for} \quad 1 \le j &\le n. \end{aligned} \qquad \circledast$$

Def Any vector $x = (x_1, \to x_n)$ that satisfies the constraints of linear problem is called a feasible solution.
An optimal solution is a feasible solution that maximizes objective function.

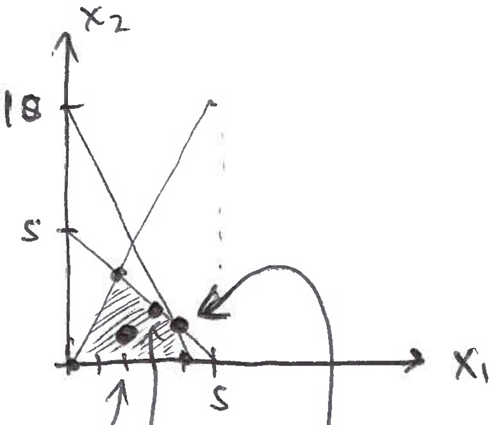
Example Maximize $z = 65x_1 + 50x_2$ subject to

$$5x_1 + 2x_2 \le 20$$

$$-2x_1 + x_2 \le 0$$

$$x_1 + x_2 \le 5$$

$$x_1 \ge 0, \quad x_2 \ge 0$$



$x = (2,1)$ Feasible solution.

$x = (3,2)$ feasible solution.

$x = \left(\frac{10}{3}, \frac{5}{3}\right)$ optimal solution (later!)

Linear Problem in Canonical Form

Find Values of $x_1, \to x_p$ that maximize $z = c_1x_1 + \cdots + c_px_p$ subject to constraints

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1p}x_p &= b_1 \\ \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mp}x_p &= b_m \end{aligned}$$

$$x_j \geq 0 \quad \text{for} \quad 1 \leq j \leq p.$$

__Claim__ Every linear problem in Standard Form can be reformulated in Canonical Form.

Consider constraint: $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$.

Introduce __Slack variable__: $u_i = b_i - a_{i1}x_1 - \cdots - a_{in}x_n$

Then $u_i \geq 0$ and $a_{i1}x_1 + \cdots + a_{in}x_n + u_i = b_i$.

Above problem ⊛ in standard form is equivalent to:
Find values of $x_1, \to x_n, u_1, \to u_m$ that will
Maximize $z = c_1x_1 + \cdots + c_nx_n$ subject to

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n + u_1 &= b_1 \\ \vdots \qquad\quad \vdots \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + u_m &= b_m \end{aligned} \qquad ⊕$$

$$x_j \geq 0 \text{ for } 1 \leq j \leq n \quad \text{and} \quad u_i \geq 0 \text{ for } 1 \leq i \leq m.$$

__Note__: If $(x_1, \to x_n, u_1, \to u_m)$ is a feasible sol. to ⊕ then $(x_1, \to x_n)$ is a $\underset{\text{optimal}}{\underline{\text{feasible}}}$ solution to ⊛.

Similarly, if $(x_1, \to x_n)$ $\underset{\text{optimal}}{\underline{\text{feasible}}}$ sol to ⊛, then get feasible sol. $(x_1, \to x_n, u_1, \to u_m)$ to ⊕ by setting $u_i = b_i - a_{i1}x_1 - \cdots - a_{in}x_n$.

Example (Hybrid car problem in canonical form.)

Maximize $z = 65x_1 + 50x_2$ subject to

$$5x_1 + 2x_2 + u_1 = 20$$
$$-2x_1 + x_2 + u_2 = 0$$
$$x_1 + x_2 + u_3 = 5$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad u_1 \geq 0, \quad u_2 \geq 0, \quad u_3 \geq 0.$$

Matrix Notation

Will use column vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1\ x_2 \cdots x_n]^T$.

Given $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ of same length, write $x \leq y$ if and only if $x_i \leq y_i$ for $1 \leq i \leq n$.

Example $\begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$ but $\begin{bmatrix} 7 \\ 5 \\ 10 \end{bmatrix} \nleq \begin{bmatrix} 8 \\ 4 \\ 11 \end{bmatrix}$

Note $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x \geq 0$ means $x_i \geq 0$ for $1 \leq i \leq n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad Ax = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} \in \mathbb{R}^m$$

$Ax \leq \mathbf{b}$ means $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{cases}$

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

Set $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$.

$$z = c_1x_1 + \cdots + c_nx_n = \vec{c} \cdot \vec{x} = [c_1 \cdots c_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c^T x.$$

Standard Form:

Find a vector $x \in \mathbb{R}^n$ that will maximize $z = c^T x$ subject to $Ax \leq b$ and $x \geq 0$.

Canonical Form:

Find a vector $x \in \mathbb{R}^n$ that will maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$.

Example (Hybrid Car, Std form)

$$A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix} \quad c = \begin{bmatrix} 65 \\ 50 \end{bmatrix}$$

Maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$.

Example (Hybrid car, canonical form)

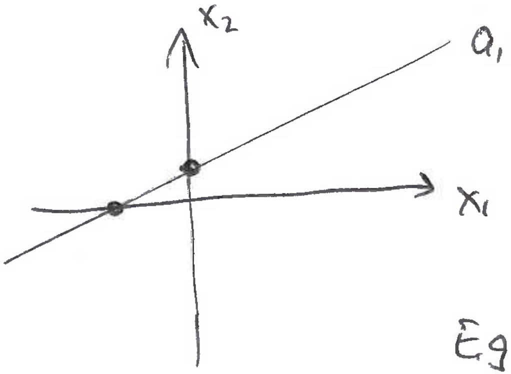
$$A' = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} \text{slack variables} \quad b = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix} \quad c = \begin{bmatrix} 65 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Maximize $c^T x$ subject to $Ax = b$ and $x \geq 0$.

Geometry of Linear Problems

Two variables: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

Equation: $a_1x_1 + a_2x_2 = b$ ⟷ line in (x_1, x_2)-plane:



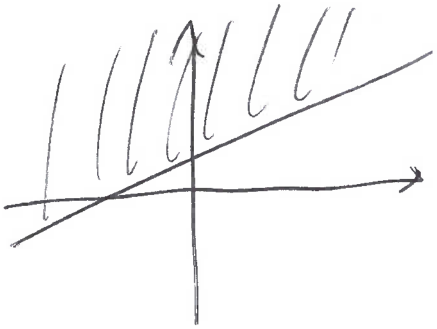
How to draw:
Find points on coordinate axes.

Eg. $x_2 = 0 \Rightarrow x_1 = \frac{b}{a_1}$.

So $(\frac{b}{a_1}, 0) \in$ line.

$(0, \frac{b}{a_2}) \in$ line.

Inequality: $a_1x_1 + a_2x_2 \le b$ ⟷ closed half-plane.



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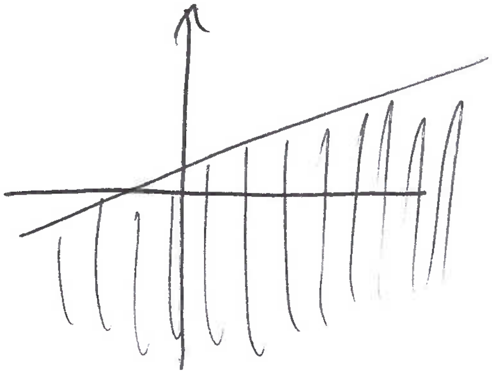


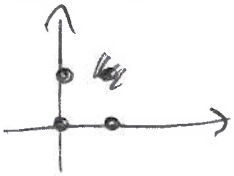
Which one?

Pick test point $(y_1, y_2) \in \mathbb{R}^2$ that is NOT on the line.

If $a_1y_1 + a_2y_2 \le b$ is satisfied: $(y_1, y_2) \in$ half-plane

Otherwise: $(y_1, y_2) \notin$ half-plane.

Good test points: $(0,0)$, $(1,0)$, $(0,1)$.



Note: one of these points is not on your line!

Example Find half-plane $2x_1 - x_2 \le 4$.

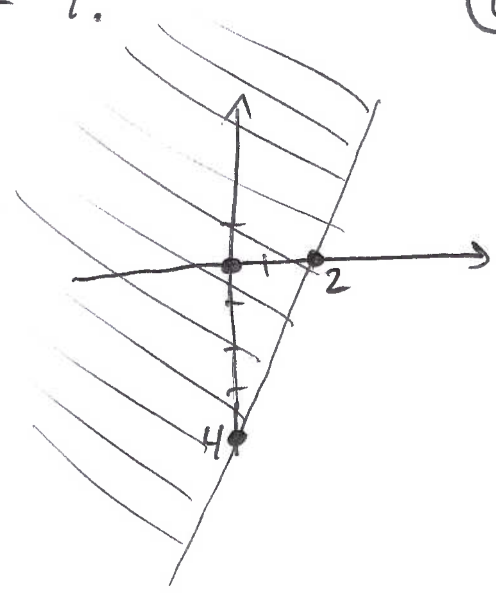
First draw line $2x_1 - x_2 = 4$.

Points on line: $(2,0)$, $(0,-4)$

Test point: $(0,0)$.

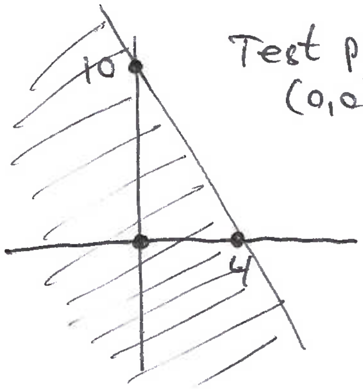
$2 \cdot 0 - 0 \le 4$ is TRUE.

$(0,0) \in$ half-plane.



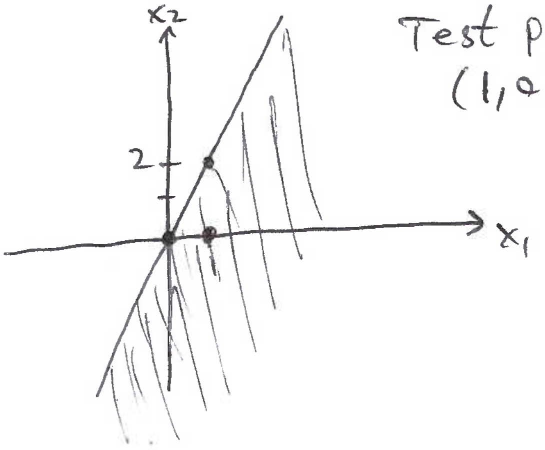
Example Constraints from car problem:

$5x_1 + 2x_2 \le 20$



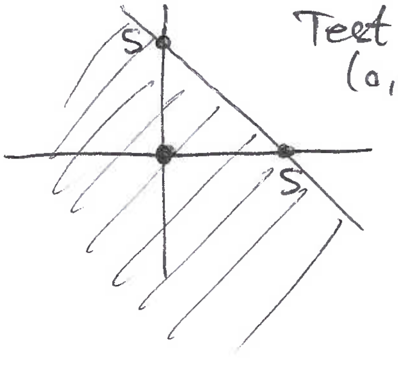
Test point $(0,0)$

$-2x_1 + x_2 \le 0$



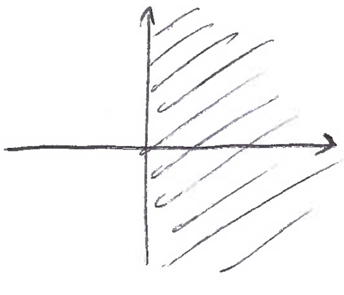
Test point: $(1,0)$

$x_1 + x_2 \le 5$

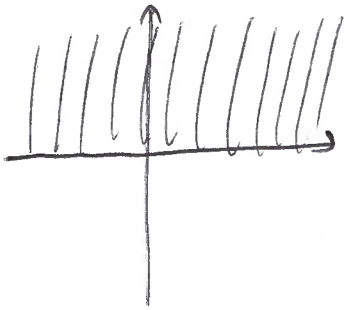


Test point $(0,0)$

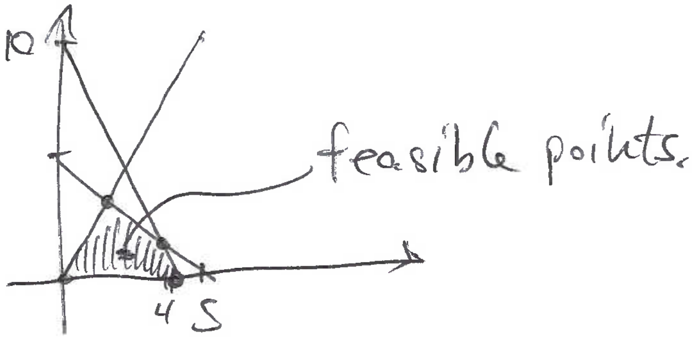
$x_1 \ge 0$



$x_2 \ge 0$

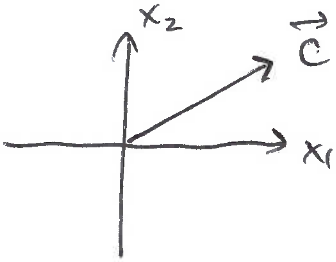


Points that sat. all constraints = intersection of all half-planes:

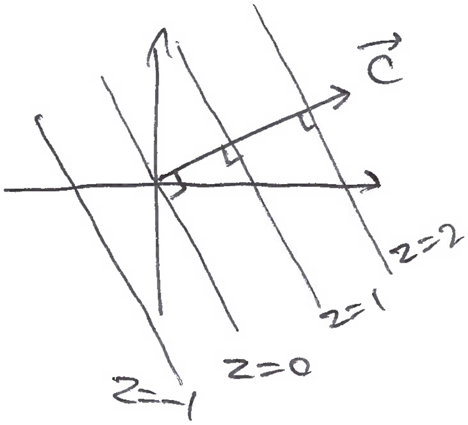


Geometry of objective function

$$z = c^T x \quad , \qquad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2 .$$



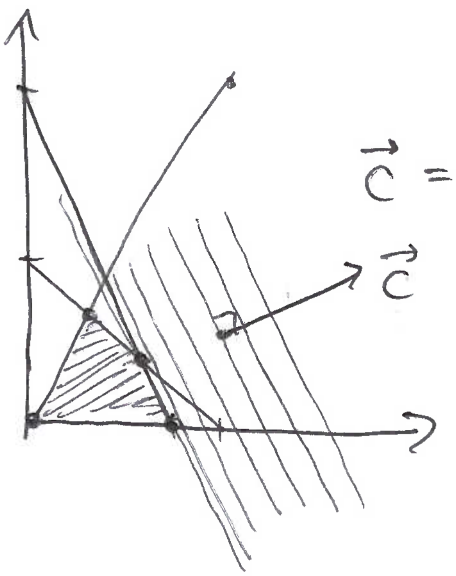
Note: For any fixed $k \in \mathbb{R}$, the equation $z = k$ or $c^T x = k$ describes a line perpendicular to $\vec{c}$.



We want to find point (x_1, x_2) within constraints that makes $c^T x$ maximal.

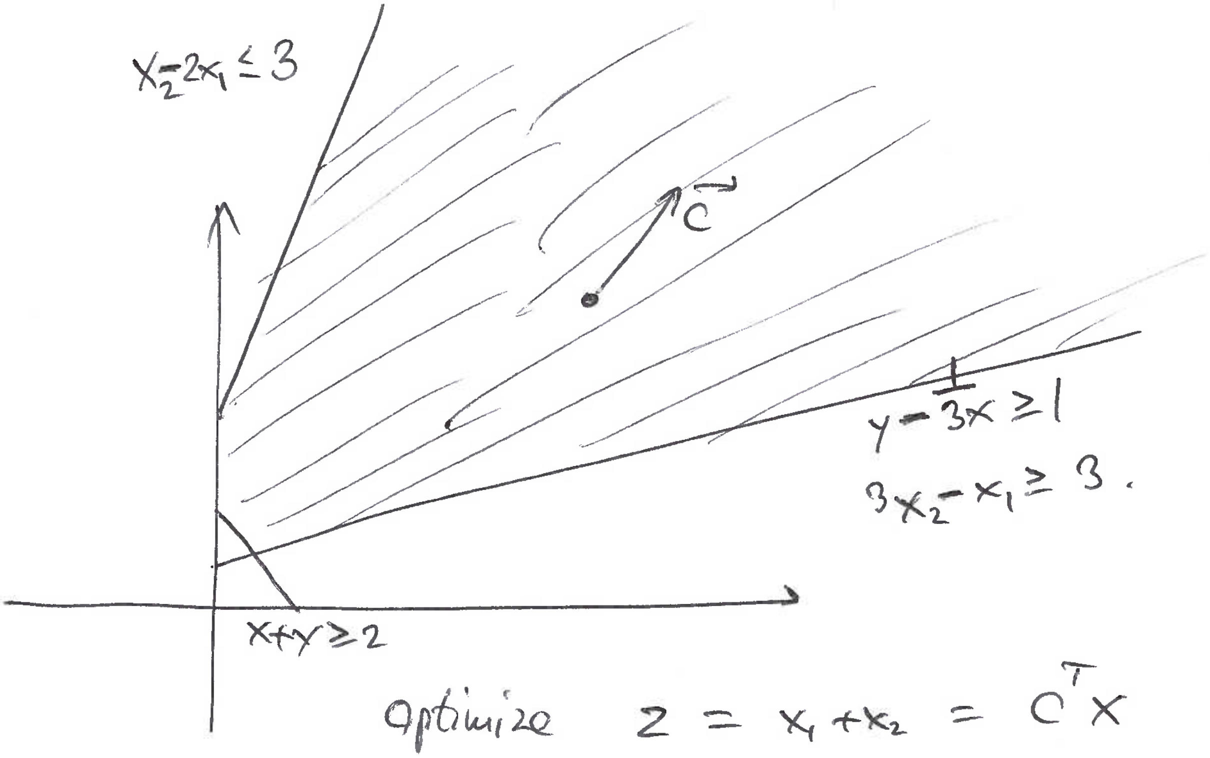
Find k such that the line $c^T x = k$ "touches" the ~~constrained the region~~ region of feasible points.

Example



$\vec{c} = (65, 50)$

Example



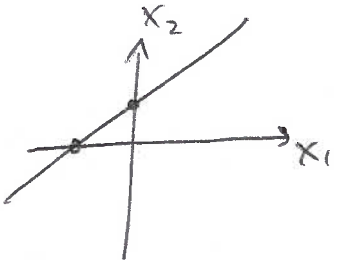
optimize $z = x_1 + x_2 = c^T x$

$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Can make z arbitrarily large.

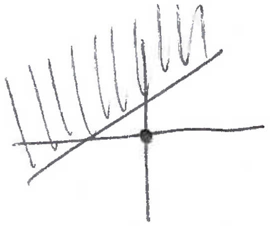
Geometry of Linear Probleme

Two variables : $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$



<u>Equation</u> : $a_1x_1 + a_2x_2 = b$ ⟷ line

<u>Inequality</u> : $a_1x_1 + a_2x_2 \le b$ ⟷ closed half-plane.



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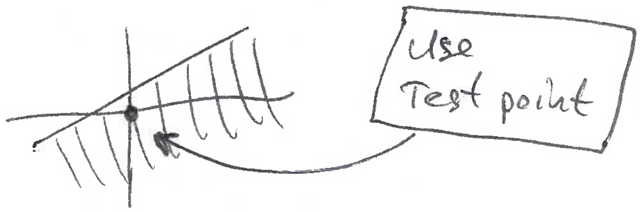
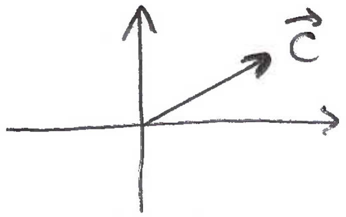

Use Test point

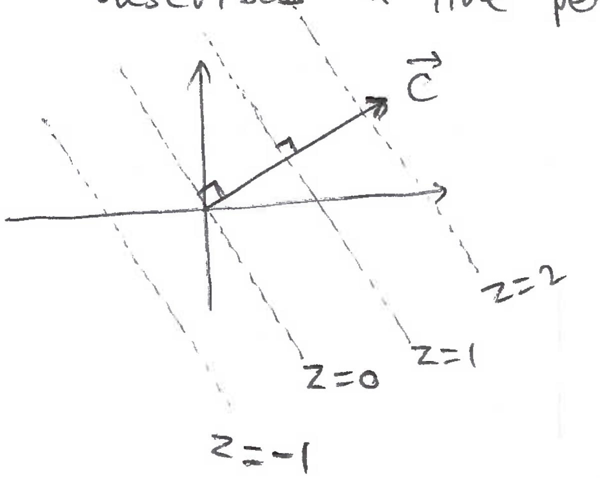
~~Intersection of constraints~~

Geometry of objective function

$$z = c_1x_1 + c_2x_2 = c^Tx \quad , \quad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2$$



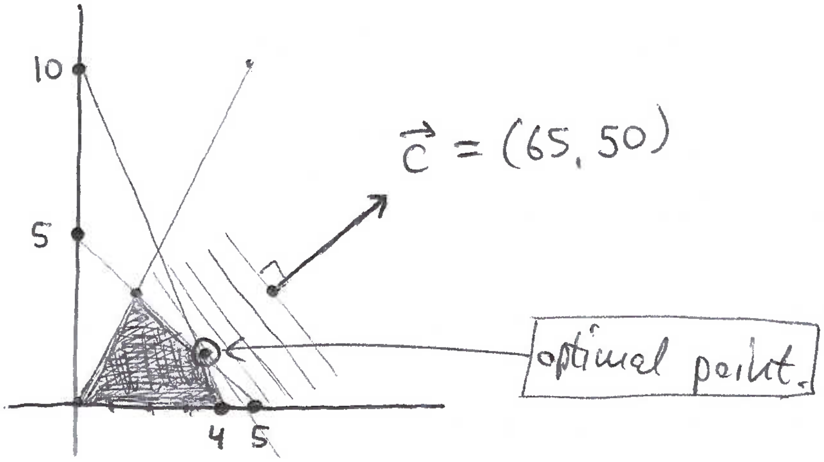
<u>Note</u> : For any fixed $k \in \mathbb{R}$, the equation $z = k$ or $c^Tx = k$ describes a line perpendicular to $\vec{c}$.



<u>Goal</u> : Find point (x_1, x_2) within constraints that makes c^Tx maximal.

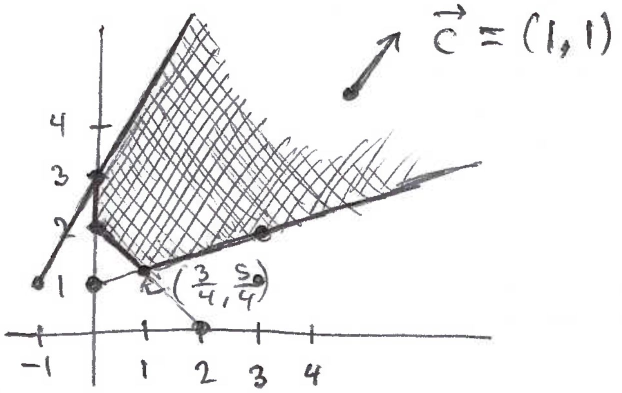
Find k such that the line $c^Tx = k$ "touches" the region of feasible points.

Example



Example Maximize $z = x_1 + x_2$ subject to constraints

$$x_2 - 2x_1 \le 3$$
$$x_1 + x_2 \ge 2$$
$$3x_2 - x_1 \ge 3$$
$$x_1 \ge 0, \quad x_2 \ge 0$$

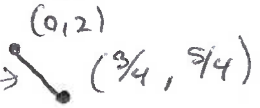


$z = c^T x = x_1 + x_2$ can become arbitrarily large.

No optimal solution.

Example Minimize $z = x_1 + x_2$ subject to same constraints.

All points on line segment →



are optimal.

Note $\vec{c}$ is <u>perpendicular</u> to side!

Example Minimize $z = x_2$ subject to same constraints.

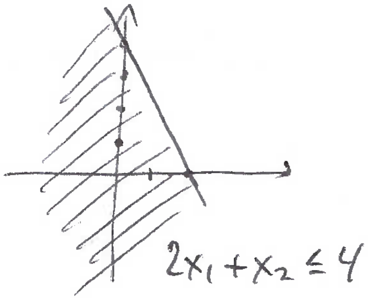
Unique optimal point: $(3/4, 5/4)$

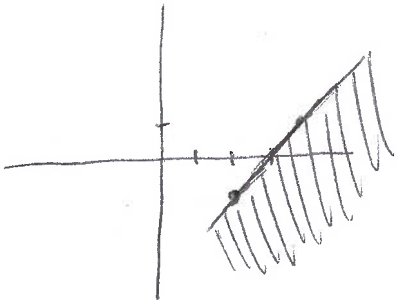
Example   Maximize $z = x_1 + 3x_2$ subject to

$2x_1 + x_2 \leq 4$

$x_1 - x_2 \geq 3$

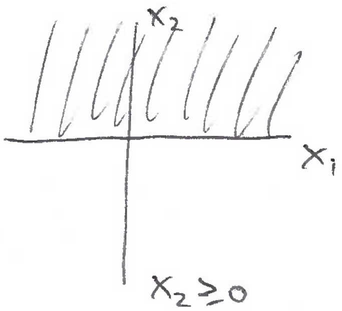
$x_1 \geq 0, \quad x_2 \geq 0$





Set of feasible points
is **EMPTY** !

No optimal point.

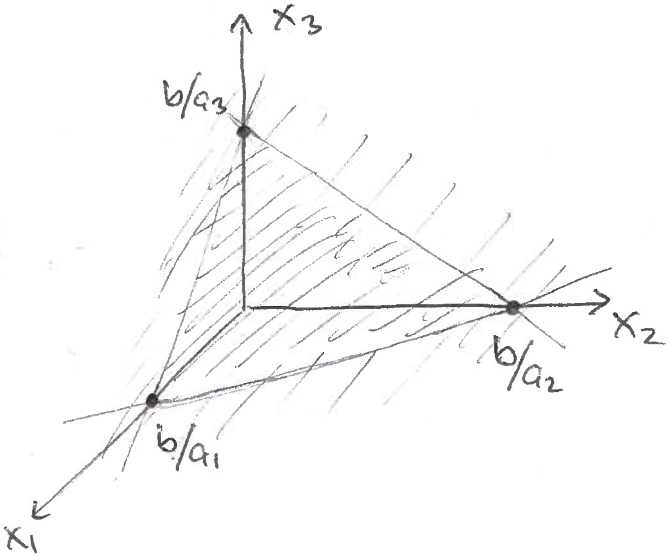


Geom of linear problems, 3 variables.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Equation $a_1x_1 + a_2x_2 + a_3x_3 = b \quad \Leftrightarrow \quad$ plane $\subseteq \mathbb{R}^3$.

Draw: Find intersections with coord. axes.



Inequality: $a_1x_1 + a_2x_2 + a_3x_3 \leq b \quad \Leftrightarrow \quad$ closed half-space of points on one side of plane.

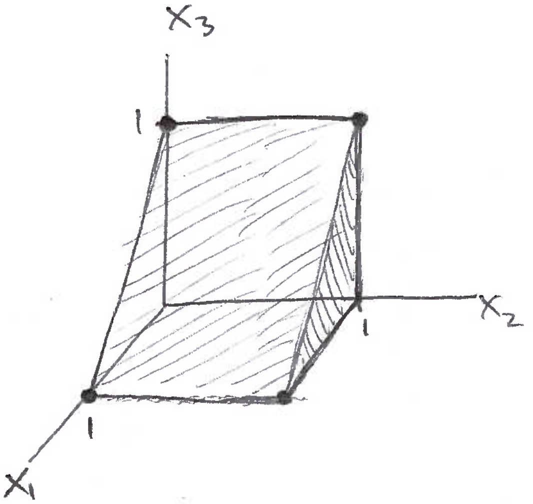
Good Test points: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Example Find points (x_1, x_2, x_3) satisfying

$$x_1 + x_3 \le 1$$

$$x_2 \le 1$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$



Note: $x_1 + x_3 = 1$ parallel to x_2-axis

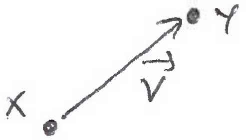
$x_2 = 1$ parallel to $x_1 x_3$-plane.

Line segments

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ be two points.

Want: line segment from x to y.

Set $\vec{v} = y - x = (y_1 - x_1, y_2 - x_2, \dots, y_n - x_n)$.



Any point on line segment given as

$$\vec{x} + \lambda \vec{v} \quad \text{for} \quad 0 \le \lambda \le 1.$$

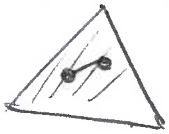
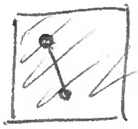
Note: $x + \lambda v = x + \lambda(y - x) = (1 - \lambda)x + \lambda y.$

Prop The line segment from x to y is the set of points

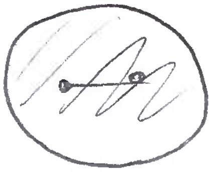
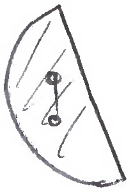
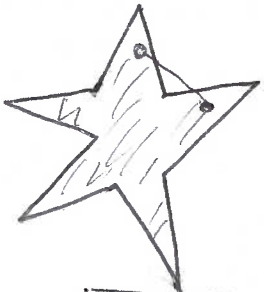
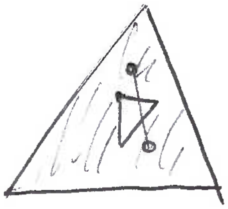
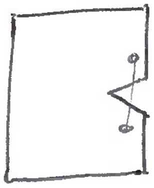
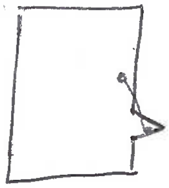
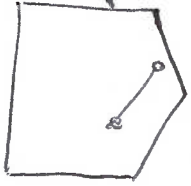
$$\{ (1-\lambda)x + \lambda y \mid 0 \le \lambda \le 1 \}$$

Def A subset $S \subseteq \mathbb{R}^n$ is convex if, whenever $x, y \in S$, the line segment from x to y is a subset of S.

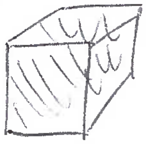
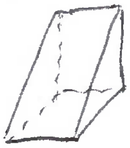
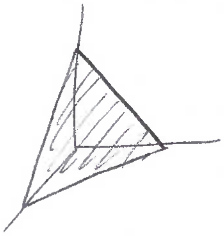
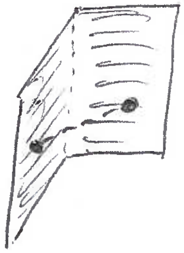
Examples In $\mathbb{R}^2$:



In $\mathbb{R}^3$:

Claim: The set of feasible points of any linear problem is a convex set.

Thm The half-~~plane~~ space in $\mathbb{R}^n$ defined by $a_1x_1 + \cdots + a_nx_n \leq b$ is convex.

Proof Write ieq in vector notation $a^Tx \leq b$, $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Let $x, y \in \mathbb{R}$ be two points in half-~~plane~~ space.

Then $a^Tx \leq b$ and $a^Ty \leq b$.

Must show that line segment from x to y is subset of half-~~plane~~ space

Let z be any point on line segment.

Then $z = (1-\lambda)x + \lambda y$ for some $\lambda \in [0,1]$.

Now $a^T z = a^T((1-\lambda)x + \lambda y)$

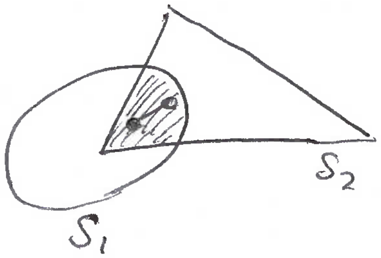
$= (1-\lambda) a^T x + \lambda a^T y$

$\leq (1-\lambda) b + \lambda b = b.$

$\therefore$ $z \in$ half-~~plane~~space.

□

Thm Let $S_1 \subseteq \mathbb{R}^n$ and $S_2 \subseteq \mathbb{R}^n$ be two convex sets. Then $S_1 \cap S_2$ is convex.

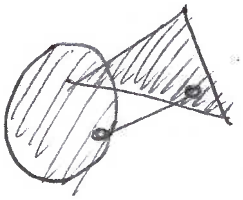


Proof Let $x, y \in S_1 \cap S_2$ be 2 pts.

Must show: line segment from x to y is a subset of $S_1 \cap S_2$.

□

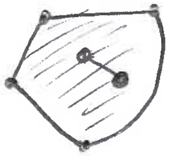
Example If S_1 and S_2 are convex, is it true that $S_1 \cup S_2$ is convex?



Cor The set of feasible points of any linear problem is convex.

Linear Opt. 9/15/2015

Def $S \subseteq \mathbb{R}^n$ is convex if, whenever $x, y \in S$, the line segment from x to y is contained in S.



Thm The half-space in $\mathbb{R}^n$ def. by $a_1x_1 + \cdots + a_nx_n \leq b$ is convex.

Thm Let $S_1 \subseteq \mathbb{R}^n$ and $S_2 \subseteq \mathbb{R}^n$ be two convex sets. Then $S_1 \cap S_2$ is convex.

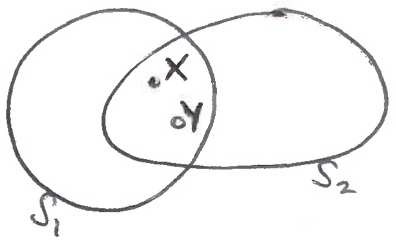
Proof Let $x, y \in S_1 \cap S_2$. Let z be a point on the line segment from x to y. Show $z \in S_1 \cap S_2$.

Since S_1 is convex, $z \in S_1$

Since S_2 is convex, $z \in S_2$.

$\therefore$ $z \in S_1 \cap S_2$.

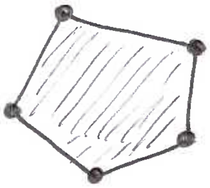
□

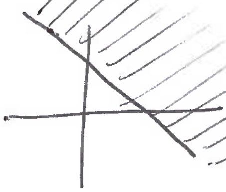


Corollary The set of feasible solutions of any linear programming problem is convex.

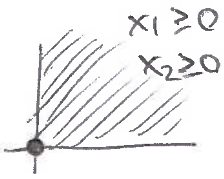
Def Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $z \in S$. We say that z is an extreme point of S if z is not an interior point of any like segment contained in S.

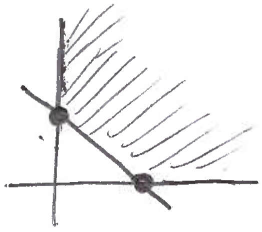
Examples





No extreme points!





Def Let $x_1, x_2, \dots, x_r \in \mathbb{R}^n$ be r points.
A convex combination of $x_1, \to x_r$ is any point of the form $z = c_1 x_1 + c_2 x_2 + \dots + c_r x_r \in \mathbb{R}^n$
where $c_1, \to c_r \in \mathbb{R}$, $\sum_{i=1}^{r} c_i = 1$, $c_i \geq 0$ for all i.

Example Let $x, y \in \mathbb{R}^n$.
A convex comb. of x, y is the same as a point on the line segment from x to y.

If $z = c_1 x + c_2 y$, $\quad c_1 + c_2 = 1$, $\quad c_1, c_2 \geq 0$:

then $z = (1-\lambda)x + \lambda y$, $\quad \lambda = c_2$, $\quad 0 \leq \lambda \leq 1$.

Thm The set S of all convex combinations of $x_1, \to x_r$ in $\mathbb{R}^n$ is convex. (S = convex hull of $x_1, \to x_r$.)

Proof Let $x, y \in S$.

Then $x = c_1 x_1 + \dots + c_r x_r$, $\quad \sum c_i = 1$, $\quad c_i \geq 0$.
$\quad\quad y = d_1 x_1 + \dots + d_r x_r$, $\quad \sum d_i = 1$, $\quad d_i \geq 0$.

Must show : For $\lambda \in [0,1]$, $(1-\lambda)x + \lambda y \in S$.
I.e. $(1-\lambda)x + \lambda y$ is a convex comb. of $x_1, \to x_r$.
Set $f_i = (1-\lambda)c_i + \lambda d_i$ for $1 \leq i \leq r$.

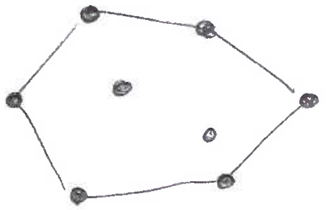
Then

- $(1-\lambda)x + \lambda y = f_1 x_1 + f_2 x_2 + \dots + f_r x_r$
- $\sum f_i = (1-\lambda)\sum c_i + \lambda \sum d_i = (1-\lambda) + \lambda = 1$
- $f_i \geq 0$ for each i.

$\square$

Example Set of convex combinations of 5 points in $\mathbb{R}^2$:

Example

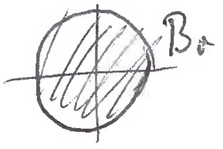


Note: The extreme points of the convex hull of $x_1, \dots, x_r$ is a subset of $\{x_1, \dots, x_r\}$.

Thm
Let $S \subseteq \mathbb{R}^n$ be convex, $z \in S$.
Then z is an extreme point of S if and only if z is not a convex combination of other points of S.

~~Extreme Point Theorem~~

For $r \in \mathbb{R}$, $r \geq 0$, define the closed ball of radius r ~~[illegible]~~ centered at origin ~~[illegible]~~ in $\mathbb{R}^n$ to be



$$B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$$ ~~$\{x \in \mathbb{R}^n$~~

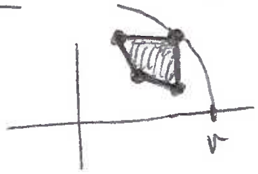
$$= \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 \leq r^2\}$$

Def A subset $S \subseteq \mathbb{R}^n$ is bounded if it is contained in some closed ball B_r.

$$(S \text{ bounded}) \iff (\exists r \in \mathbb{R}_+ : S \subseteq B_r).$$

Example Let $y_1, y_2, \dots, y_k \in \mathbb{R}^n$ be k points.
Then the convex hull of $y_1, \dots, y_k$ is bounded.

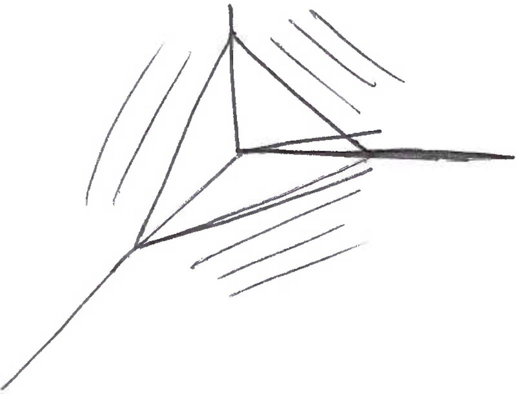
Exercise: Contained in B_r where $r = \max\{\|y_1\|, \dots, \|y_k\|\}$



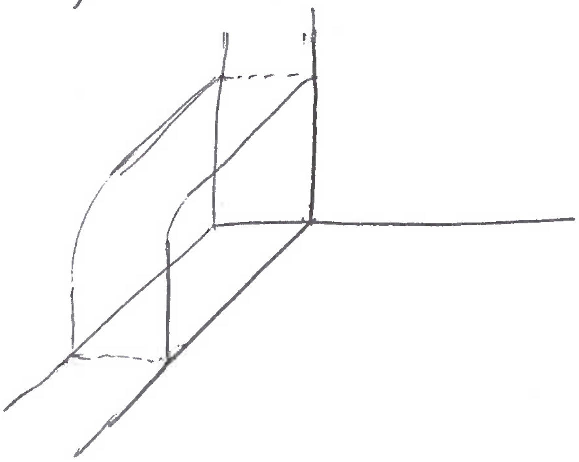
Def An intersection of half-spaces in $\mathbb{R}^n$ is called a *convex polytope*.

Examples

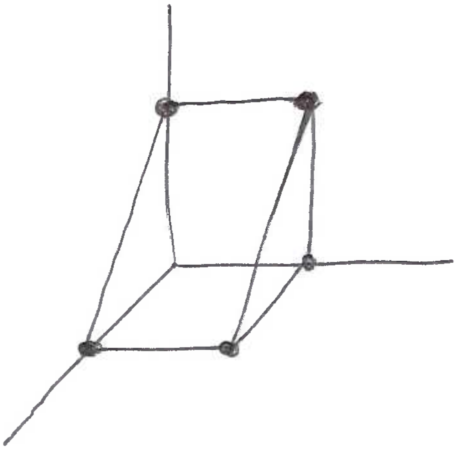
1) $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$:



2) $x \geq 0,\ x_2 \leq 1$ in $\mathbb{R}^3$:

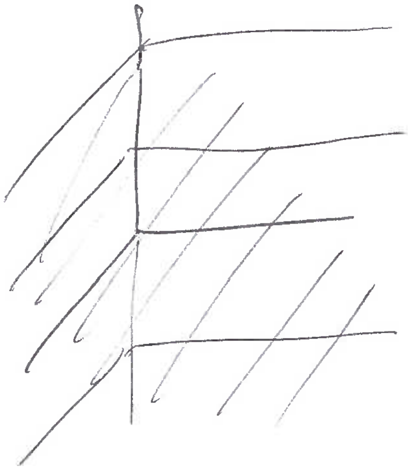


3) $x \geq 0,\ x_2 \leq 1,$
$x_1 + x_3 \leq 1.$



Fact: A convex polytope has an extreme point if and only if it does NOT contain a line.

Example $x_1, x_2 \geq 0$:



No extreme points.

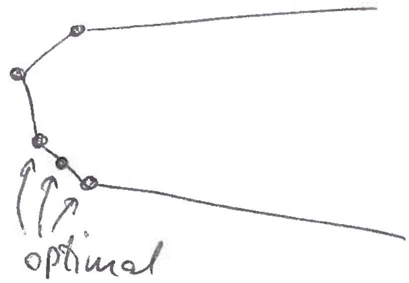
Extreme Point Theorem

Consider a general linear programming problem.
Let $S \subseteq \mathbb{R}^n$ be the set of feasible solutions.

(1) If S is non-empty and bounded, then an optimal solution exists and occurs at an extreme point.

(2) If S non-empty and unbounded, and if an optimal solution exists, [<u>And</u> S has at least one extr. pt.] then an optimal solution occurs at an extreme point.

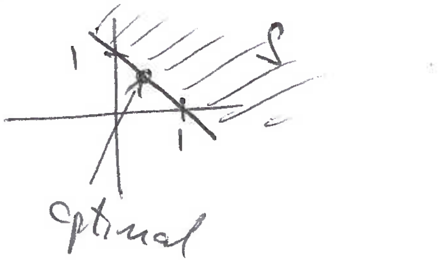
(3) If an optimal solution does not exist, then S is empty OR S is unbounded.



Q: Which part is FALSE ?

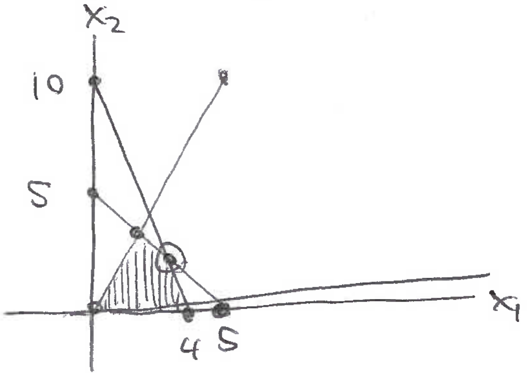
<u>Example</u> Find $(x_1, x_2) \in \mathbb{R}^2$ such that $z = x_1 + x_2$ is maximal subject to $x_1 + x_2 \leq 1$.

- An optimal solution exists, but S has no extreme points.



Example Maximize $z = 6x_1 + 5x_2$ subject to

$$5x_1 + 2x_2 \le 20$$
$$2x_1 - x_2 \ge 0$$
$$x_1 + x_2 \le 5$$
$$x_1 \ge 0, \quad x_2 \ge 0$$



S = set of feasible solutions

bounded.

Extreme points:

$(0,0)$	$z = 0$
$(4,0)$	$z = 24$
$\left(\frac{5}{3}, \frac{10}{3}\right)$	$z = \frac{80}{3}$
$\left(\frac{10}{3}, \frac{5}{3}\right)$	$z = \frac{85}{3}$

optimal point!

Linear Opt. 9/18/2015

A $m \times s$ matrix. $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$.

Consider sytem of linear equations $Ax = b$, $x \in \mathbb{R}^n$.

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{is}x_s = b_i \quad \text{for } 1 \le i \le m$$

Row i of A : $[a_{i1}, a_{i2}, \cdots, a_{is}] = R_i$ $\qquad R_i^T x = b_i$

Assume rows of A are <u>not</u> linearly independent.

Say row R_m is a linear combination of $R_1, \to R_{m-1}$.

$R_m = t_1 R_1 + \cdots + t_{m-1} R_{m-1}$, $t_i \in \mathbb{R}$

Then $R_m^T x = t_1 R_1^T x + t_2 R_2^T x + \cdots + t_{m-1} R_{m-1}^T x$

Two possibilities :

1) $b_m \ne t_1 b_1 + t_2 b_2 + \cdots + t_{m-1} b_{m-1}$:

The linear system is inconsistent, $Ax = b$ is NOT possible. No solutions.

2) $b_m = t_1 b_1 + t_2 b_2 + \cdots + t_{m-1} b_{m-1}$:

The linear system is consistent.

$R_m^T x = b_m$ follows from $R_i^T x = b_i$, $1 \le i \le m-1$.

~~[illegible]~~ So equation m is <u>redundant</u>.

Can replace A with $A' = \begin{bmatrix} R_1 \\ \vdots \\ R_{m-1} \end{bmatrix}$, b with $b' = \begin{bmatrix} b_1 \\ \vdots \\ b_{m} \end{bmatrix}$

<u>Conclude</u> : Any $\overbrace{}^{\text{consistent}}$ linear system $Ax = b$ is equivalent to a linear system $A'x = b'$ where A' is an $m' \times s$ matrix with m' linearly independent rows.

Example

$$A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 4 & 4 & 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 20 \\ 0 \\ 5 \\ 25 \end{bmatrix}$$

Note:
Rows of A NOT
linearly indep.

$$Ax = b \quad , \qquad x \in \mathbb{R}^5$$

$$5x_1 + 2x_2 + x_3 = 20$$ ~~4/20~~

$$-2x_1 + x_2 + x_4 = 0$$

$$x_1 + x_2 + x_5 = 5$$

$$4x_1 + 4x_2 + x_3 + x_4 + x_5 = 25$$

Last equation is sum of 3 others, redundant.

Replace A with first 3 rows, b with $\begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix}$.

Note: If b had been $\begin{bmatrix} 20 \\ 0 \\ 5 \\ 27 \end{bmatrix}$ then $Ax = b$ would be inconsistent!

Consider now linear system $Ax = b$, A $m \times s$ matrix with m linear independent rows.

<u>Note</u>:
- $m \leq s$. Otherwise rows cant be lin. indep. in $\mathbb{R}^n$.
- $\operatorname{rank}(A) = m = \dim(\text{row space})$

Let $A_1, A_2, \ldots, A_s \in \mathbb{R}^m$ be the columns of A.

$$A = \left[\begin{array}{c|c|c|c|c} A_1 & A_2 & A_3 & \cdots & A_s \end{array}\right]$$

<u>Note</u>: $m = \operatorname{rank}(A) = \dim(\text{col space})$

$\Rightarrow$ It is possible to choose m linearly independent columns $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$.

And: These columns form a basis for $\mathbb{R}^m$.

<u>Note</u>: $Ax = b$ is equivalent to:

$$x_1 A_1 + x_2 A_2 + \cdots + x_s A_s = b.$$

<u>Note</u>: If $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ are linearly independent columns, then

$$x_{i_1} A_{i_1} + x_{i_2} A_{i_2} + \cdots + x_{i_m} A_{i_m} = b$$

has a unique solution in $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$.

Example

$$A = \begin{array}{ccccc} \left[\begin{array}{c|c|c|c|c} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \end{array} \qquad b = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix} \qquad x \in \mathbb{R}^5 .$$

Columns A_2, A_3, A_5 are linearly independent.

$x_2 A_2 + x_3 A_3 + x_5 A_5 = b$ has unique solution.

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix}$$

$$\left.\begin{array}{lcl} 2x_2 + x_3 & = & 20 \\ x_2 & = & 0 \\ x_2 + 0 + x_5 & = & 5 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x_3 = 20 \\ x_2 = 0 \\ x_5 = 5 \end{array}\right.$$

Note: $x = (0, 0, 20, 0, 5)^T$ is a solution to $Ax = b$.

~~Def (Basic Solution)~~

~~Let A be an $m \times s$ matrix of rank m, and let $b \in \mathbb{R}^m$. A basic solution to the system $Ax = b$ is any solution $x \in \mathbb{R}^s$ with the property:~~

~~Let $i_1 \leq i_2 < \cdots \leq i_k$ be the indices for which x_{i_t}~~

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix} \in \mathbb{R}^s$.

The support of x is the set of indices i for which $x_i \neq$

support(x) = $\{ i \in \mathbb{Z} \mid 1 \leq i \leq s \text{ and } x_i \neq 0 \}$.

Example support of $(3, 0, 0, 1, -3, 0, 2)^T$ is $\{1, 4, 5, 7\}$.

Def Let A be an $m \times s$ matrix of rank m, $b \in \mathbb{R}^m$.
A basic solution to $Ax = b$ is any solution $x \in \mathbb{R}^s$ such that the cols. of A corresponding to $\text{supp}(x)$ are linearly independent.

I.e. if $\text{supp}(x) = \{i_1 < i_2 < \cdots < i_k\}$ then
$A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are lin. indep. vectors in $\mathbb{R}^m$.

Consider linear problem in canonical form:

A $m \times s$ matrix. $b \in \mathbb{R}^m$. $c \in \mathbb{R}^s$. $\text{rank}(A) = m$.

Maximize $z = c^T x$ subject to

$$Ax = b$$
$$x \geq 0$$

A basic feasible solution is a basic sol. to $Ax = b$ that is also feasible, i.e. $x \geq 0$.

Thm Let $\overset{\subset \mathbb{R}^s}{S}$ be the set of all feasible solutions.
The extreme points of S are exactly the basic feasible solutions.

Proof

Let $x \in S$ be an extreme point.

Let $\text{supp}(x) = \{i_1, i_2, \ldots, i_k\}$.

$$\begin{cases} x_j > 0 & \text{for } j \in \text{supp}(x) \\ x_j = 0 & \text{for } j \notin \text{supp}(x) \end{cases}$$

Show $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are linearly independent.

Assume $t_{i_1} A_{i_1} + t_{i_2} A_{i_2} + \dots + t_{i_k} A_{i_k} = 0 \in \mathbb{R}^m$

for some $t_{i_1}, t_{i_2}, \to t_{i_k} \in \mathbb{R}$, NOT all zero.

~~Def. $t \in \mathbb{R}^s$ by setti~~

Def. $t = (t_1, t_2, \to t_s)^T \in \mathbb{R}^s$ by setting $t_j = 0$ for $j \notin \text{supp}(x)$.

Then $t \neq 0$ and $At = 0$.

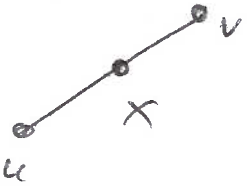
Therefore $A(x + dt) = b$ for all $d \in \mathbb{R}$.

Since $t_j \neq 0 \Rightarrow x_j > 0$, we can choose $d > 0$ so small that

$$u = x + dt \geq 0 \quad \text{in } \mathbb{R}^s$$
$$v = x - dt \geq 0 \quad \text{in } \mathbb{R}^s.$$

But then $u, v \in S$ and $x = \frac{1}{2}u + \frac{1}{2}v$ is on line segment from u to v. ↯.

So x was not an extreme point.



Now let $x \in \mathbb{R}^n$ be any basic feasible solution.

Then $Ax = b$, $x \geq 0$, so $x \in S$.

Show x extreme point of S.

otherwise choose $u, v \in S$, $u \neq v$, $x = \frac{1}{2}u + \frac{1}{2}v$

<u>Note</u>: since $u \geq 0$, $v \geq 0$, have
$\text{Supp}(u - v) \subseteq \text{Supp}(u + v) = \text{Supp}(x)$.

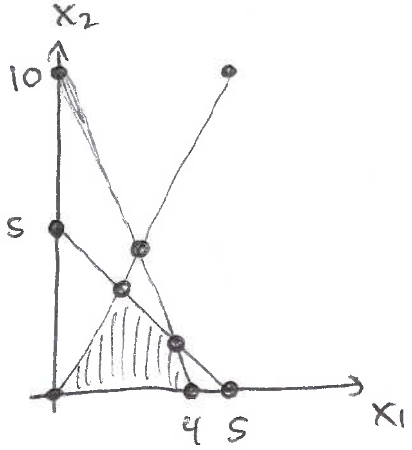
Since $A(u - v) = Au - Av = b - b = 0$, columns of A corresp. to $\text{supp}(u - v)$ are NOT lin. indep.
$\Rightarrow x$ NOT basic sol. ↯.

Example $A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} \in \mathbb{R}^5.$

Maximize $z = 65x_1 + 50x_2$ subject to $Ax = b$, $x \geq 0$.

This is canonical form of Hybrid Car Problem.

(x_3, x_4, x_5 are slack variables.)



Basic Solutions to $Ax = b$:

$$x_1 A_1 + x_2 A_2 + x_3 A_3 = b$$

$$\left.\begin{aligned} 5x_1 + 2x_2 + x_3 &= 20 \\ -2x_1 + x_2 &= 0 \\ x_1 + x_2 &= 5 \end{aligned}\right\} \begin{aligned} x_1 &= \tfrac{5}{3} \\ x_2 &= \tfrac{10}{3} \\ x_3 &= 5 \end{aligned}$$

$$x = \left(\tfrac{5}{3}, \tfrac{10}{3}, 5, 0, 0\right)^T$$

$$x_1 A_1 + x_2 A_2 + x_4 A_4 = b$$

$x_1 = \frac{10}{3}$ $x_2 = \frac{5}{3}$, $x_4 = 5$

$$x = \left(\tfrac{10}{3}, \tfrac{5}{3}, 0, 5, 0\right)^T$$

$$x_1 A_1 + x_2 A_2 + x_5 A_5 = b$$

$x = \left(\frac{20}{9}, \frac{40}{9}, 0, 0, -\frac{5}{3}\right)$ %

NOT FEASIBLE !!

$x = (5, 0, -5, 10, 0)$ %

$x = (0, 0, 20, 0, 5)$

$x = (4, 0, 0, 8, 1)$

$x = (0, 5, 10, -5, 0)$ %

$x = (0, 10, 0, -10, -5)$ %

Linear Opt. 9/22/2015 ①

Standard Form vs. Canonical Form

Std. Form: $(n=2)$ $x \in \mathbb{R}^2$

Maximize $z = 2x_1 + x_2$

subject to

$x_1 + x_2 \leq 1$

$x_1 \geq 0$, $x_2 \geq 0$

Canonical Form: $(s=3)$ $x \in \mathbb{R}^3$

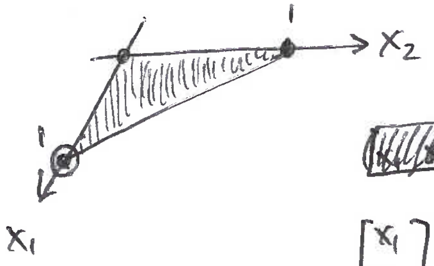
(x_3 slack variable.)

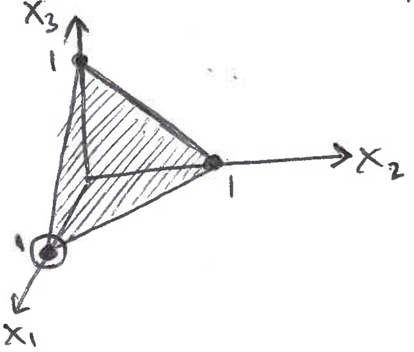
Maximize $z = 2x_1 + x_2$

subject to

$x_1 + x_2 + x_3 = 1$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$





$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \longmapsto \begin{bmatrix}x_1\\x_2\\1-x_1-x_2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \longleftarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Problem P (Std Form):

$x \in \mathbb{R}^n$, m inequalities.

Maximize $z = C^T x = c_1 x_1 + \cdots + c_n x_n$

subject to $Ax \leq b$, $x \geq 0$

A $m \times n$ matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

$S \subseteq \mathbb{R}^n$ set of feasible sols.

Problem P' (Canonical Form):

$x \in \mathbb{R}^s$, $s = n+m$.

Maximize $z = c_1 x_1 + \cdots + c_n x_n$

subject to $A'x = b$, $x \geq 0$

$A' = \begin{array}{|c|c|}\hline A & I \\\hline\end{array}$ $m \times s$ matrix

$S' \subseteq \mathbb{R}^s$ set of feasible sols.

Def $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^s$

$$\begin{array}{|c|}\hline x \\\hline\end{array} \longmapsto \begin{array}{|c|}\hline x \\\hline \\\hline\end{array} \!\!- b - Ax$$

Bijection: $\mathbb{R}^n \xrightarrow[\cong]{\phi} \{x \in \mathbb{R}^s \mid A'x = b\}$

$\phi(x)$ satisfies $A'\phi(x) = b$:

$$A'\phi(x) = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ b-Ax \end{bmatrix} = Ax + I(b-Ax) = b$$

Inverse map: $\pi : \mathbb{R}^s \longrightarrow \mathbb{R}^n$, $\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_s \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Bijection:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, leftrightarrow] & \{x \in \mathbb{R}^s \mid A'x = b\} \\
S \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow] & S' \arrow[u, phantom, sloped, "\subseteq"] \\
\left\{\begin{matrix}\text{extreme} \\ \text{points of } S\end{matrix}\right\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{extreme} \\ \text{points of } S'\end{matrix}\right\} \arrow[u, phantom, sloped, "\subseteq"] \\
& \left\{\begin{matrix}\text{basic feasible} \\ \text{solutions to } P'\end{matrix}\right\} \arrow[u, phantom, sloped, "="]
\end{tikzcd}$$

Write $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$

R_i = i-th row of A.

Note: $Ax \le b \iff \begin{cases} R_1 x \le b_1 \\ R_2 x \le b_2 \\ \vdots \\ R_m x \le b_m \end{cases}$

Thm Let $x \in S \subseteq \mathbb{R}^n$ be a feasible solution to P. Then x is an extreme point of S if and only if x satisfies n independent equalities from the list:

$$\{R_1 x = b_1,\ R_2 x = b_2, \dots, R_m x = b_m,\ x_1 = 0,\ x_2 = 0, \dots, x_n = 0\}.$$

Example (Hybrid Car)

Standard Form: $x \in \mathbb{R}^2$

$$A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix}$$

Maximize $z = 6x_1 + 5x_2$

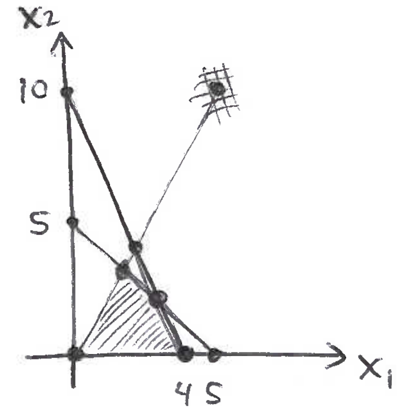
Subject to $Ax \leq b$, $x \geq 0$

Canonical Form: $x \in \mathbb{R}^5$

$$A' = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = \boxed{A_1 \mid A_2 \mid A_3 \mid A_4 \mid A_5}$$

Maximize $z = 6x_1 + 5x_2$

Subject to $A'x = b$, $x \geq 0$.



Basic Solutions to $Ax = b$:

① $x_1 A_1 + x_2 A_2 + x_3 A_3 = b$

$$\left.\begin{aligned} 5x_1 + 2x_2 + x_3 &= 20 \\ -2x_1 + x_2 &= 0 \\ x_1 + x_2 &= 5 \end{aligned}\right\} \quad \begin{aligned} x_1 &= \tfrac{5}{3} \\ x_2 &= \tfrac{10}{3} \\ x_3 &= 5 \end{aligned}$$

$x = \left(\frac{5}{3}, \frac{10}{3}, 5, 0, 0\right)^T$

② $x_1 A_1 + x_2 A_2 + x_4 A_4 = b$

$x_1 = \frac{10}{3}$, $x_2 = \frac{5}{3}$, $x_4 = 5$

$x = \left(\frac{10}{3}, \frac{5}{3}, 0, 5, 0\right)^T$

③ $x_1 A_1 + x_2 A_2 + x_5 A_5 = b$

$x = \left(\frac{20}{9}, \frac{40}{9}, 0, 0, -\frac{5}{3}\right)^T$ Not feasible!

④ $x = (5, 0, -5, 10, 0)^T$ N.F.

⑤ $x = (0, 0, 20, 0, 5)^T$

⑥ $x = (4, 0, 0, 8, 1)^T$

⑦ $x = (0, 5, 10, -5, 0)$ N.F.

⑧ $x = (0, 10, 0, -10, -5)$ N.F.

Extreme Point Thm

(i) If set of feasible points is non-empty and bounded, then an optimal solution exists and occurs at an extreme point.

Extreme points : $z = 6x_1 + 5x_2$

$(0,0)$	$z = 0$
$(4,0)$	$z = 24$
$(\frac{5}{3}, \frac{10}{3})$	$z = \frac{80}{3}$
$(\frac{10}{3}, \frac{5}{3})$	$z = \frac{85}{3}$

$(\frac{10}{3}, \frac{5}{3})$ optimal solution !

Simplex Method Ideas

Let $P \subseteq \mathbb{R}^n$ be a convex polyhedron.

A face of P is a non-empty subset $F \subseteq P$ that is the set of optimal points for some objective fcn $z = c^T x$.

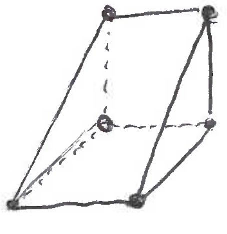
Equivalently, F is the intersectin of P with a half-space $c^T x \geq z_0$ for which P is contained in the opposite half-space $c^T x \leq z_0$. (We allow $c = 0$, so that P is a face of itself.)

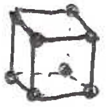
Examples

0-dim. faces are called vertices or extreme points

Faces of codim. 1 are called facets.

1-dim. faces are like segments between vertices that are contained in boundary of P.





Def Two vertices of P are adjacent if the line segment between them is a 1-dim. face.

Simplex Method Given objective fcn $z = c^T x$ on polyh. P

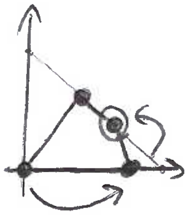
Start with any vertex x_0 of P.

If some adjacent vertex x_1 satisfies $c^T x_1 > c^T x_0$,

then replace x_0 with x_1.

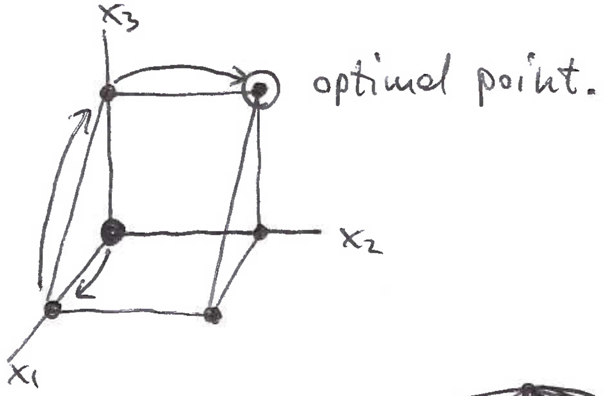
Repeat until $c^T x_1 \leq c^T x_0$ for all adjacent vertices x_1.

Example $z = 6x_1 + 5x_2$



Example Maximize $z = x_1 + 2x_2 + 3x_3$

Subject to $x_2 \leq 1$, $x_1 + x_3 \leq 1$, $x \geq 0$



Example

[redacted]
Relatively few extreme points on this path!

Row Operations

$$A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 20 \\ 0 \\ 5 \end{bmatrix}$$

Maximize $z = c^T x$
Subject to $Ax = b, \; x \geq 0$

$$c^T = [6 \;\; 5 \;\; 0 \;\; 0 \;\; 0]$$

$$\cancel{A' = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & -1 & 0 \end{bmatrix}}$$

$$A' = \begin{bmatrix} 5 & 2 & 1 & 0 & 1 \\ -2 & 1 & 0 & 1 & 0 \\ 6 & 3 & 1 & 0 & 1 \end{bmatrix} \qquad b' = \begin{bmatrix} 20 \\ 0 \\ 25 \end{bmatrix}$$

Max. $z = c^T x$
Subj. to $A'x = b'$

$$c'^T = [11 \;\; 7 \;\; 1 \;\; 0 \;\; 0]$$

Max $z' = c'^T x$
Subj. to $A'x = b'$.

Consider linear system of equations

$$Ax = b$$

A $m \times s$ matrix, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^s$.

Assume A has rank m.

$$A = \boxed{A_1 \mid A_2 \mid \cdots \mid A_s}$$

Recall: Basic solution: Choose $1 \le i_1 < i_2 < \cdots < i_m \le s$ such that $A_{i_1}, A_{i_2}, \cdots, A_{i_m}$ indep. in $\mathbb{R}^m$.

Solve $x_{i_1} A_{i_1} + x_{i_2} A_{i_2} + \cdots + x_{i_m} A_{i_m} = 0$.

Set $x_j = 0$ for $j \notin \{i_1, \cdots, i_m\}$.

$x = (x_1, x_2, \cdots, x_s)^T \in \mathbb{R}^s$ basic solution.

Basic variables: $\{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\}$.

Def A basic solution x is non-degenerate if all basic variables $x_{i_k} \neq 0$.

Def Let $x, y \in \mathbb{R}^s$ be basic solutions, with basic variables $\{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\}$ and $\{y_{j_1}, y_{j_2}, \cdots, y_{j_m}\}$. We say that x and y are adjacent basic solutions if $\{i_1, i_2, \cdots, i_m\}$ and $\{j_1, j_2, \cdots, j_m\}$ have $m-1$ indices in common. ~~[illegible]~~

Note: We may have $x = y$. But if x or y is non-degenerate then $x \neq y$.

Today: Will assume all basic sols. to $Ax = b$ are non-degenerate.
Equivalent: b is not a linear combination of any collection of $m-1$ columns of A.

Example

$$A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 1 \\ 5 \end{bmatrix} \quad x \in \mathbb{R}^5 \qquad Ax = b.$$

obvious basic solution : $x = (0, 0, 20, 1, 5)^T$

$\boxed{\text{basic vars: } x_3, x_4, x_5}$

Find adjacent basic sol : Use basic vars x_2, x_4, x_5 :

$$x_2 A_2 + x_4 A_4 + x_5 A_5 = b$$

$$\left\{ \begin{aligned} 2x_2 + 0x_4 + 0x_5 &= 20 \\ 1x_2 + 1x_4 + 0x_5 &= 1 \\ 1x_2 + 0x_4 + 1x_5 &= 5 \end{aligned} \right\} \qquad \begin{aligned} x_2 &= 10 \\ x_4 &= -9 \\ x_5 &= -5 \end{aligned}$$

Adjacent basic sol : $x = (0, 10, 0, -9, -5)$

Note: If $[A|b]$ row equivalent to $[A'|b']$, then $A'x = b'$ has same solutions and same basic solutions!

Idea: Rewrite $Ax = b$ so that $x = (0, 10, 0, -9, -5)$ is "the obvious solution".

$$\left[\begin{array}{ccccc|c} 5 & \textcircled{2} & 1 & 0 & 0 & 20 \\ -2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccccc|c} \frac{5}{2} & 1 & \frac{1}{2} & 0 & 0 & 10 \\ -2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 5 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccccc} & x_2 & & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{5}{2} & 1 & \frac{1}{2} & 0 & 0 & 10 \\ -\frac{9}{2} & 0 & -\frac{1}{2} & 1 & 0 & -9 \\ -\frac{3}{2} & 0 & -\frac{1}{2} & 0 & 1 & -5 \end{array}\right] \end{array}$$

$\boxed{\textcircled{2} \text{ pivot}}$

Terminology: Going from $(0,0,20,1,5)$ to $(0,10,0,-9,-5)$

x_2 is entering basic variable.

x_3 is departing basic variable.

Find adjacent feasible ~~solution~~ basic solution:

Have basic feasible solution $x = (0,0,20,1,5)$ with basic vars. $\{x_3, x_4, x_5\}$

This means $A_3, A_4, A_5 \in \mathbb{R}^3$ lin. indep.

First: Choose new column, say A_2.

Corresponding entering variable: x_2.

New solution will look like

$$x_2 A_2 + x_3 A_3 + x_4 A_4 + x_5 A_5 = b$$

with x_3, x_4, or x_5 equal to zero.

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \\ 5 \end{bmatrix}$$

$$2x_2 + x_3 = 20$$
$$x_2 + x_4 = 1$$
$$x_2 + x_5 = 5$$

If we change x_2, ~~to positive~~ then x_3, x_4, x_5 also change. Must remain ≥ 0 !!

$$20 - 2x_2 = x_3 \geq 0 \Rightarrow x_2 \leq 10$$
$$1 - x_2 = x_4 \geq 0 \Rightarrow x_2 \leq 1$$
$$5 - x_2 = x_5 \geq 0. \Rightarrow x_2 \leq 5$$

$\therefore$ Can increase x_2 to 1 ; x_4 will decrease to 0.

$$[A|b]=\begin{array}{c} \\ \\ \\ \end{array}\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 5 & 2 & 1 & 0 & 0 & 20 \\ -2 & \textcircled{1} & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 5 \end{array}\right]$$

Entering variable: x_2

Departing variable: x_4

$$\sim \left[\begin{array}{ccccc|c} 9 & 0 & 1 & -2 & 0 & 18 \\ -2 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & -1 & 1 & 4 \end{array}\right]$$

New "obvious" basic variables:

$\{x_2, x_4, x_5\}$

New basic solution:

$$x = (0, 1, 18, 0, 4)^T.$$

Example

Maximize $z = 6x_1 + 5x_2$ subject to

$$Ax = b, \quad A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 1 \\ 5 \end{bmatrix} \quad x \in \mathbb{R}^5.$$

Note: $x_1 + x_2 + x_5 = 5$

$$z = 6x_1 + 5x_2 + (5 - x_1 - x_2 - x_5)$$

$$= 5x_1 + 4x_2 - x_5 + 5$$

Same objective function!!

Idea:

Given some basic feasible sol. to $Ax = b$;

Rewrite z as $z = c_1x_1 + c_2x_2 + \cdots + c_5x_5$

Such that $c_i = 0$ whenever x_i basic variable!

Use obvious basic solution $x = (0, 0, 20, 1, 5)^T$

Basic vars x_3, x_4, x_5.

Note $z = 6x_1 + 5x_2$ already on this form!

Point of this:

If $c_i > 0$ for some i, then $x_i = 0$ NOT basic.

We can increase $z = c_1x_1 + \cdots + c_sx_s$ by increasing x_i, letting x_i be entering variable. Must decrease some leaving variable x_j. We can do this for free since $c_j = 0$!!!

Rewrite $z = c_1x_1 + \cdots + c_sx_s$ as an additional eqn:

$$-c_1x_1 - c_2x_2 - \cdots - c_sx_s + z = 0.$$

$(m+1) \times (s+1)$:

$$A' = \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -c_1 \ -c_2 \ \cdots \ -c_s & 1 \end{array}\right] \qquad b' = \left[\begin{array}{c} b \\ \hline 0 \end{array}\right]$$

~~Given [illegible]~~

Tableaus: Given linear problem and some basic feasible solution x with basic variables $\{x_{i_1}, \ldots, x_{i_m}\}$, organize information in Tableau:

Names

	x_1	x_2	$\cdots$	x_s	z	
x_{i_1}	a_{11}	a_{12}	$\cdots$	a_{1s}	0	b_1
x_{i_2}	a_{21}	a_{22}		a_{2s}	0	b_2
$\vdots$	$\vdots$					
x_{i_m}	a_{m1}	a_{m2}		a_{ms}	0	b_m
	$-c_1$	$-c_2$		$-c_m$	1	0

Demand: Column i_k has a 1 in the row marked by x_{i_k}, zeros in all other entries.

Example

$$
\begin{array}{c|cccccc|c}
 & x_1 & \overset{\downarrow}{x_2} & x_3 & x_4 & x_5 & z & \\
\hline
x_3 & 5 & 2 & 1 & 0 & 0 & 0 & 20 \\
\leftarrow x_4 & -2 & \textcircled{1} & 0 & 1 & 0 & 0 & 1 \\
x_5 & 1 & 1 & 0 & 0 & 1 & 0 & 5 \\
\hline
 & -6 & -5 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Entering variable x_2.

Departing variable x_4

Row ops give:

$$
\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & z & \\
\hline
x_3 & 9 & 0 & 1 & -2 & 0 & 0 & 18 \\
x_2 & -2 & 1 & 0 & 1 & 0 & 0 & 1 \\
x_5 & 3 & 0 & 0 & -1 & 1 & 0 & 4 \\
\hline
 & -16 & 0 & 0 & 5 & 0 & 1 & 5
\end{array}
$$

Linear Opt. 9/29/2015

Example Maximize $z = 6x_1 + 5x_2$ subject to
$Ax = b$, $x \geq 0$, where $A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 20 \\ 1 \\ 5 \end{bmatrix}$ $x \in \mathbb{R}^5$.

Start with obvious BFS: $x = (0, 0, 20, 1, 5)^T$

Basic variables: $\{x_3, x_4, x_5\}$

Want: Adjacent BFS s.t. $z = 6x_1 + 5x_2$ is larger.

Recall: To obtain adjacent BFS, choose (non-basic) entering variable, increase it, justify basic variables.

Right now: $z = 6x_1 + 5x_2 = 0$.

We can choose to increase either x_1 or x_2.

Entering variable: x_1

New BFS: $x_1 A_1 + x_3 A_3 + x_4 A_4 + x_5 A_5 = b$ (with x_3 or x_4 or $x_5 = 0$.

$$5x_1 + x_3 = 20 \Rightarrow 20 - 5x_1 = x_3 \geq 0 \Rightarrow x_1 \leq \theta_1 = \frac{20}{5} =$$

$$-2x_1 + x_4 = 1 \Rightarrow 1 + 2x_1 = x_4 \geq 0 \Rightarrow x_1 \geq \theta_2 = -\frac{1}{2}$$

$$x_1 + x_5 = 5 \Rightarrow 5 - x_1 = x_5 \geq 0 \Rightarrow x_1 \leq \theta_3 = \frac{5}{1} = 5$$

Increase x_1 to 4. Departing variable: x_3

$$\left[\begin{array}{ccccc|c} \textcircled{5} & 2 & 1 & 0 & 0 & 20 \\ -2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & \frac{2}{5} & \frac{1}{5} & 0 & 0 & 4 \\ -2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & \frac{2}{5} & \frac{1}{5} & 0 & 0 & 4 \\ 0 & 9/5 & 2/5 & 1 & 0 & 9 \\ 0 & 3/5 & -\frac{1}{5} & 0 & 1 & 1 \end{array}\right]$$

$x = (4, 0, 0, 9, 1)$ New obvious ~~solution~~ BFS.

Basic vars
x_1, x_4, x_5.

Now: $z = 6x_1 + 5x_2 = 6 \cdot 4 = 24.$

Can we do better?

PROBLEM: ~~If we try to~~

We can try to increase x_2.

BUT x_1 is now a basic variable, so we may have to decrease x_1 !!

So not clear if z will get larger/smaller.

IDEA: $x_1 + \frac{2}{5}x_2 + \frac{1}{5}x_3 = 4$ for all feasible $x \in \mathbb{R}^5$.

$$\Rightarrow \quad z = 6x_1 + 5x_2$$

$$= 6x_1 + 5x_2 + \lambda\left(4 - x_1 - \frac{2}{5}x_2 - \frac{1}{5}x_3\right)$$

for any $z \in \mathbb{R}$.

Rewrite z as linear comb. of non-basic variables x_2, x_3!

Take $\lambda = 6$.

$$z = 6x_1 + 5x_2 + 24 - 6x_1 - \frac{12}{5}x_2 - \frac{6}{5}x_3$$

$$= \frac{13}{5}x_2 - \frac{6}{5}x_3 + 24$$

Current obvious BFS: $x = (4, 0, 0, 9, 1)^T$

If we increase x_2 then z becomes larger.

(x_3 does not change!)

If we increase x_3 then z becomes smaller.

(x_2 does not change.)

Entering variable : x_2

New BFS : $x_1 A_1' + x_2 A_2' + x_4 A_4' + x_5 A_5' = b'$

$$\begin{aligned} x_1 + \tfrac{2}{5}x_2 &= 4 \\ \tfrac{9}{5}x_2 + x_4 &= 9 \\ \tfrac{3}{5}x_2 + x_5 &= 1 \end{aligned}$$

$$x_1 = 4 - \tfrac{2}{5}x_2 \quad \Rightarrow \quad x_2 \le \theta_1 = \frac{4}{2/5} = 10$$

$$x_4 = 9 - \tfrac{9}{5}x_2 \quad \Rightarrow \quad x_2 \le \theta_2 = \frac{9}{9/5} = 5$$

$$x_5 = 1 - \tfrac{3}{5}x_2 \quad \Rightarrow \quad x_2 \le \frac{1}{3/5} = 5/3$$

Increase x_2 to $\frac{5}{3}$.

Departing variable : x_5.

$$\left[\begin{array}{ccccc|c} 1 & \frac{2}{5} & \frac{1}{5} & 0 & 0 & 4 \\ 0 & \frac{9}{5} & \frac{2}{5} & 1 & 0 & 9 \\ 0 & \textcircled{\tfrac{3}{5}} & -\frac{1}{5} & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & \frac{10}{3} \\ 0 & 0 & 1 & 1 & -3 & 6 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{5}{3} & \frac{5}{3} \end{array}\right]$$

New obvious BFS : $x = (\frac{10}{3}, \frac{5}{3}, 0, 6, 0)^T$

Basic vars : x_1, x_2, x_4.

Rewrite z as linear comb. of non-basic vars : x_3, x_5

$$z = \cancel{2}\ \cancel{\tfrac{13}{5}x_2} - \frac{6}{5}x_3 + 24 + \frac{13}{5}\left(\frac{5}{3} - \cancel{x_2} + \frac{1}{3}x_3 - \frac{5}{3}x_5\right)$$

$$= [illegible] - \frac{1}{3}x_3 - \frac{13}{3}x_5 + \frac{85}{3}$$

<u>Now</u> : If we increase x_3 then z <u>decreases</u> !
If we incr. x_5 then z <u>decreases</u>. !

$\therefore$ $x = (\frac{10}{3}, \frac{5}{3}, 0, 6, 0)^T$ optimal sol.

Simplex Algorithm

At each step given:

- a BFS $x \in \mathbb{R}^s$
- A system of constraints $Ax = b$, $x \geq 0$ so that the BFS is an obvious solution.
- $z = c^T x$ expressed as linear combination of non-basic variables. + constant.

~~Express this information in a tableau: a_{11} a_{12} . . .~~

Trick: Treat $z = c_1x_1 + c_2x_2 + \cdots + c_sx_s + z_0$ as an additional equation:

$$-c_1x_1 - c_2x_2 - \cdots - c_sx_s + z = z_0$$

Express this information in a tableau:

	x_1	x_2	x_3	$\cdots$	x_s	z	
x_{i_1}	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1s}	0	b_1
x_{i_2}	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2s}	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_{i_m}	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{ms}	0	b_m
	$-c_1$	$-c_2$	$-c_3$	$\cdots$	$-c_s$	1	z_0

Label columns by variables.

Note: Columns of basic variables contain one 1, m zeros.

Label row i by basic variable whose column contains 1 in row i.

Initial tableau:

Consider std problem:

Maximize $z = c_1 x_1 + \cdots + c_n x_n$

subject to $Ax \leq b$, $x \geq 0$.

A $m \times n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$.

Assumption: $b \geq 0$ (so that $x = 0$ is a feasible sol.)

Slack variables: $x_{n+1}, x_{n+2}, \ldots, x_s$, $s = n+m$.

$$A' = \boxed{A \;\big|\; I_m}$$

Max $z = c_1 x_1 + \cdots + c_n x_n$ subj. to

$A'x = b$, $x \geq 0$.

This system has obvious BFS: $x = (0, 0, \ldots, 0, x_{n+1}, \ldots, x_s)$

$= (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m$

Initial tableau:

	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_s	z	
x_{n+1}	a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	0	b_1
x_{n+2}	a_{21}	a_{22}	$\cdots$	a_{2n}	0	1	$\cdots$	0	0	b_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_s	a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	0	b_m
	$-c_1$	$-c_2$	$\cdots$	$-c_n$	0	0	$\cdots$	0	1	0

Check optimality: If some entry in bottom row, say $-c_i$ is negative, then we can increase z by increasing x_i (and we can keep all other non-basic vars at 0.)

If all entries in bottom row are ≥ 0, then the current obvious BFS is optimal.

⑥

Step of Simplex Algorithm:

If not optimal: choose column number j with negative entry in bottom row.

Want to increase x_j

Entering variable x_j.

Form θ-ratios: $\theta_i = \frac{b_i}{a_{ij}}$ for $1 \leq i \leq m$.

If $\theta_i > 0$: The requirement $x_k \geq 0$ implies $x_j \leq \theta_i$.

If $\theta_i < 0$: The requirement $x_k \geq 0$ implies $x_j \geq \theta_i$ (ignore this!)

(If $\theta_i = \frac{b_i}{0}$: $a_{ij} = 0$, x_i can't be departing variable because cols. $A_j, A_{m+1}, \ldots, \widehat{\cancel{A}}_i, \ldots, A_s$ not lin. indep.)

If $\theta_i < 0$ for all i:

We can increase x_j arbitrarily.

$\Rightarrow$ can increase z arbitrarily

$\therefore$ NO OPTIMAL SOLUTION!

If $\theta_i > 0$ for at least one i:

Choose i so that θ_i has minimal positive value.

Departing variable: basic variable of row i.

Do row operations so that tableau gets pivot at (i,j)-entry.

⑦

Same example with tableaux:

	$\downarrow x_1$	x_2	x_3	x_4	x_5	z	
$\leftarrow x_3$	$\bigcirc\!\!\!\!\!5$	2	1	0	0	0	20
x_4	−2	1	0	1	0	0	1
x_5	1	1	0	0	1	0	5
	−6	−5	0	0	0	1	0

Enter: x_1

$\theta_1 = \frac{20}{5} = 4$

$\theta_2 = \frac{1}{-2}$

$\theta_3 = \frac{5}{1} = 5.$

Depart: x_3

	x_1	$\downarrow x_2$	x_3	x_4	x_5	z	
x_1	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0	4
x_4	0	$9/5$	$2/5$	1	0	0	9
$\leftarrow x_5$	0	$\bigcirc\!\!\!\!\!\!\!3/5$	$-\frac{1}{5}$	0	1	0	1
	0	$-\frac{13}{5}$	$\frac{6}{5}$	0	0	1	24

Enter: x_2

$\theta_1 = \frac{4}{2/5} = 10$

$\theta_2 = \frac{9}{9/5} = 5$

$\theta_3 = \frac{1}{3/5} = \frac{5}{3}$

Depart: x_5

	x_1	x_2	x_3	x_4	x_5	z	
x_1	1	0	$1/3$	0	$-2/3$	0	$10/3$
x_4	0	0	1	+[illegible]	−3	0	6
x_2	0	1	$-1/3$	0	$5/3$	0	$5/3$
	0	0	$1/3$	0	$13/3$	1	$85/3$

optimal solution: $x = (\frac{10}{3}, \frac{5}{3}, 0, 6, 0)$

Max value of z : $\frac{85}{3}$.

Linear Opt. 10/2/2015

Std. Problem: Maximize ~~[illegible]~~ $z = c_1x_1 + \cdots + c_nx_n$

subj. to $Ax \le b$, $x \ge 0$.

Assume: $b \ge 0$.

Initial tableau:

	x_1	x_2	$\cdots$	x_n	x_{n+1}		$\cdots$	x_s	z	
x_{n+1}	a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	0	b_1
x_{n+2}	a_{21}	a_{22}	$\cdots$	a_{2n}	0	1	$\cdots$	0	0	b_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_s	a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	0	b_m
	$-c_1$	$-c_2$	$\cdots$	$-c_n$	0	0	$\cdots$	0	1	0

~~Check optimal~~ Current BFS: $x = (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m)^T$

Check Optimality:

- If some entry in ~~bottom~~ objective row is negative, say $-c_j < 0$, then we can increase z by increasing x_i.
  Not optimal!
- If all entries in ~~bottom~~ objective row are ≥ 0, then current BFS is optimal.

Step of Simplex Algorithm:

If not optimal: Choose column # j with negative entry in objective row.

Entering variable: x_j

Compute θ-ratios: $\theta_i = \dfrac{b_i}{a_{ij}}$ for $1 \le i \le m$.

If $\theta_i > 0$: Requirement $x_k \ge 0 \Rightarrow x_j \le \theta_i$ k = basic var of row i.

If $\theta_i < 0$: Requirement $x_k \ge 0 \Rightarrow x_j \ge \theta_i$

If $\theta_i = \frac{b_i}{0}$: $a_{ij} = 0$, No restriction on x_j from $x_k \ge 0$.

If $\theta_i < 0$ or undef. for all i:

We can increase x_j arbitrarily $\Rightarrow$ can increase z arbitrarily.

$\therefore$ NO OPTIMAL SOLUTION !!

If $\theta_i > 0$ for at least one i:

Choose i s.t. θ_i has smallest positive value.

Pivot: (i,j)-entry of tableau.

Departing variable: Basic variable of row i.

Do row operations to make col. j a "01-column" with 1 at (i,j).

Example: Maximize $z = 6x_1 + 5x_2$ subject to

$$5x_1 + 2x_2 \leq 20$$
$$-2x_1 + x_2 \leq 1$$
$$x_1 + x_2 \leq 5$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

	↓ x_1	x_2	x_3	x_4	x_5	z	
← x_3	$\boxed{5}$	2	1	0	0	0	20
x_4	-2	1	0	1	0	0	1
x_5	1	1	0	0	1	0	5
	-6	-5	0	0	0	1	0

$\theta_1 = \frac{20}{5} = 4$

$\theta_2 = \frac{1}{-2} = -\frac{1}{2}$

$\theta_3 = \frac{5}{1} = 5$

Enter: x_1

Pivot: $(1,1)$

Depart: x_3

	x_1	↓ x_2	x_3	x_4	x_5	z	
x_1	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0	4
x_4	0	$\frac{9}{5}$	$\frac{2}{5}$	1	0	0	9
← x_5	0	$\boxed{\frac{3}{5}}$	$-\frac{1}{5}$	0	1	0	1
	0	$-\frac{13}{5}$	$-\frac{1}{5}$	0	0	1	24

$\theta_1 = \frac{4}{2/5} = 10$

$\theta_2 = \frac{9}{9/5} = 5$

$\theta_3 = \frac{1}{3/5} = \frac{5}{3}$

Enter: x_2

Pivot: $(3,2)$

Depart: x_5

	x_1	x_2	x_3	x_4	x_5	z	
x_1	1	0	$1/3$	0	$-2/3$	0	$10/3$
x_4	0	0	1	1	-3	0	6
x_2	0	1	$-1/3$	0	$5/3$	0	$5/3$
	0	0	$1/3$	0	$13/3$	1	$85/3$

Optimal solution :

$x = (\frac{10}{3}, \frac{5}{3}, 0, 6, 0)$

Max value of z : $\frac{85}{3}$.

Observations :

- At each step the non-zero entries of current BFS are in last column.
- z(current BFS) = bottom-right entry.
- Column of z-variable never changes. Can be dropped.

Degenerate Solutions & Cycling (sect. 2.2)

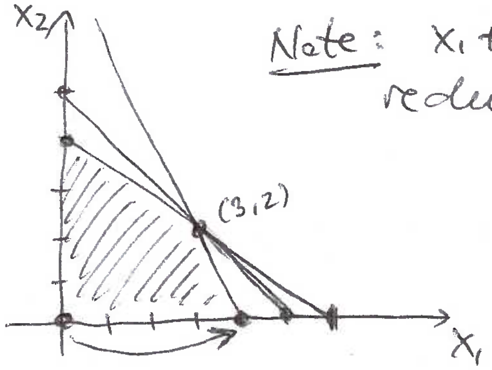
Example Maximize $z = x_1 + 6x_2$ subject to

$2x_1 + x_2 \le 8$

$2x_1 + 3x_2 \le 12$

$x_1 + x_2 \le 5$

$x_1 \ge 0, \quad x_2 \ge 0$



Note: $x_1 + x_5 \le 5$ redundant

	↓ x_1	x_2	x_3	x_4	x_5	
← x_3	②	1	1	0	0	8
x_4	2	3	0	1	0	12
x_5	1	1	0	0	1	5
	-1	-6	0	0	0	0

$\Theta_1 = \frac{8}{2} = 4$

$\Theta_2 = \frac{12}{2} = 6$

$\Theta_3 = \frac{5}{1} = 5$

BFS : $(0,0,8,12,5)$

Enter: x_1

Pivot: (1,1)

Depart: x_3

	x_1	↓ x_2	x_3	x_4	x_5	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	4
x_4	0	2	-1	1	0	4
← x_5	0	$\textcircled{\scriptsize 1/2}$	$-1/2$	0	1	1
	0	$-11/2$	$1/2$	0	0	4

$\Theta_1 = 8$

$\Theta_2 = 2$

$\Theta_3 = 2$

BFS : $(4,0,0,4,1)$

Enter: x_2

Pivot: (3,2)

Depart: x_5

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & \overset{\downarrow}{x_3} & x_4 & x_5 & \\ \hline x_1 & 1 & 0 & 1 & 0 & -1 & 3 \\ \leftarrow x_4 & 0 & 0 & \textcircled{1} & 1 & -4 & 0 \\ x_2 & 0 & 1 & -1 & 0 & 2 & 2 \\ \hline & 0 & 0 & -5 & 0 & 11 & 15 \end{array}$$

$\Theta_1 = 3$

$\Theta_2 = 0$

$\Theta_3 = -2$

Enter: x_3

Pivot: $(2,3)$

Depart: x_4

BFS: $x = (3, 2, 0, 0, 0)$

is <u>degenerate</u>!

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline x_1 & 1 & 0 & 0 & -1 & 3 & 3 \\ x_3 & 0 & 0 & 1 & 1 & -4 & 0 \\ x_2 & 0 & 1 & 0 & 1 & -2 & 2 \\ \hline & 0 & 0 & 0 & 5 & -9 & 15 \end{array}$$

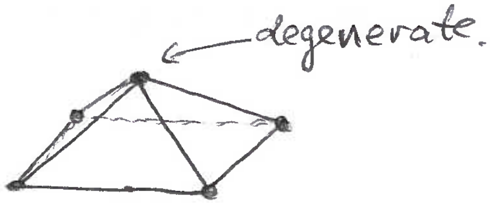
BFS: $x = (3, 2, 0, 0, 0)$

$$\boxed{\text{Same solution, different basic variables.}}$$

Suppose we could choose x_4 as entering variable here.
Would get back to previous solution.

Such examples exist.

<u>Remark</u>: If 2-dim. polyhedron has degenerate vertex, then some inequality is redundant. NOT true for 3-dim. polyhedron:



Bland's Rule for choosing pivot:

1) Entering variable x_j:

Choose j to be the smallest column with a negative entry in objective row.

2) Departing variable x_k:

If several rows are tied with same minimal positive θ_i, choose the row i for which the corresponding basic variable x_k has smallest possible index k.

Thm When Bland's Rule is used, no cycles will happen, and the Simplex Algorithm will produce an optimal solution or show that none exists.

Bland's Rule:

1) Select entering variable:

Choose x_j where j smallest column with negative entry in obj. row.

2) Select departing variable:

If several rows are tied with same minimal θ_i, choose row i for which corresp. basic variable x_k has ~~minimal~~ smallest possible index k.

Thm When Bland's Rule is used, no cycling will happen and Simplex Alg. will produce an optimal solution or show that none exists.

Example Choose pivot with Bland's Rule

	x_1	x_2	x_3	↓ x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}		
x_1	1	−2	4	−2	1	0	0	0	0	−1	0	3	$\theta_1 = -\frac{3}{2}$
x_8	0	0	1	3	2	0	1	1	0	2	0	1	$\theta_2 = ⓘ1/3$
← x_6	0	−3	0	⑨	0	1	−5	0	0	−2	0	3	$\theta_3 = 1/3$ ←
x_9	0	2	3	0	0	0	1	0	1	0	0	2	$\theta_4 = \infty$
x_{11}	0	−1	−1	6	2	0	3	0	0	0	1	2	$\theta_5 = 1/3$
	0	3	0	−2	1	0	−4	0	0	−1	0	15	

Enter: x_4 Depart: x_6

Linear Opt. 10/9/2015

Artificial Variables: Finding initial BFS.

Recall: Every linear problem can be solved by solving a problem in canonical form.

Example Minimize $z = c_1x_1 + c_2x_2 + c_3x_3$
subject to constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &\ge b_3 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Introduce stack variables $u_1, u_2 \ge 0$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + u_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - u_2 &= b_3 \\ x_1, x_2, x_3, u_1, u_2 &\ge 0. \end{aligned}$$

$\boxed{\text{Maximize } -z = \sum_{i=1}^{3} -c_ix_i}$

OBS Consider Canonical problem: Maximize $z = c^Tx$ subj. to. $Ax = b$, $x \ge 0$, A $m\times s$ matrix, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^s$.

Suppose $b_i < 0$ for some i.

~~Replace b_i and i-th row of A~~

Switch sign of b_i and i-th row of A.

E.g. $3x_1 + x_2 - 4x_3 = -2$ $\leftarrow$ $\boxed{\text{i-th row}}$

Change to $-3x_1 - x_2 + 4x_3 = 2$.

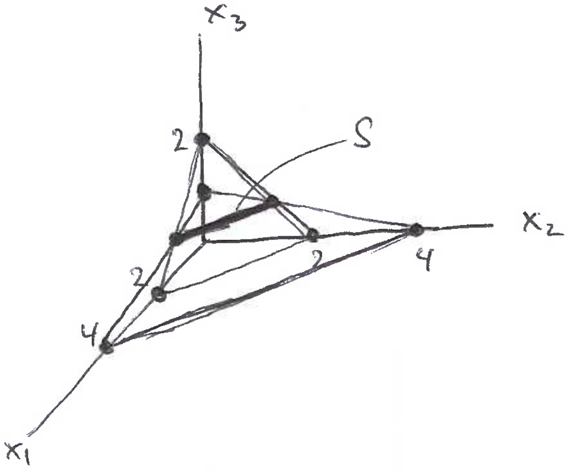
$\therefore$ Every linear problem in canonical form (Max $z = c^T x$ subj to $Ax = b$, $x \geq 0$) can be written so that $\underline{b \geq 0}$ in $\mathbb{R}^m$.

<u>Goal</u>: Solve such problems with simplex method.

<u>Problem</u>: If problem does not come from problem in Std. form, then it is NOT CLEAR how to find initial basic feasible solution.

Or even: Is set of feasible solutions $S \subseteq \mathbb{R}^s$ empty or not?

<u>Example</u> $S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 = 2 \\ x_1 + x_2 + 4x_3 = 4 \\ x_1, x_2, x_3 \geq 0 \end{array} \right\}$

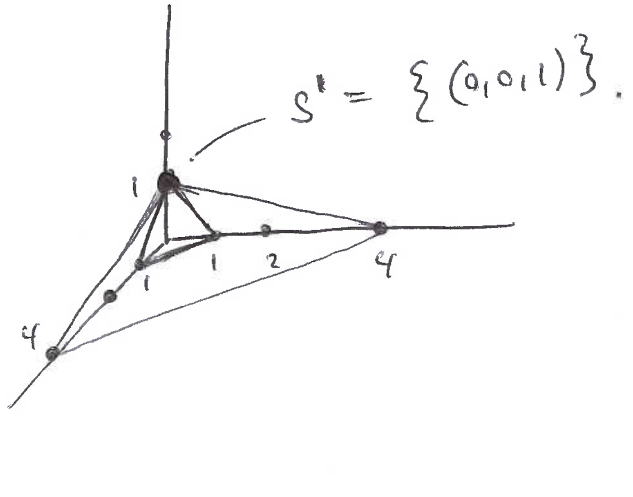


S': $\begin{array}{l} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + 4x_3 = 4 \\ x_1, x_2, x_3 \geq 0 \end{array}$

S'': $\begin{array}{l} x_1 + x_2 + x_3 = \frac{1}{2} \\ x_1 + x_2 + 4x_3 = 4 \\ x \geq 0 \end{array}$

$S'' = \emptyset$

$\left(\begin{array}{l} 3x_3 = 3.5 \\ \Rightarrow 4x_3 > 4 \end{array} \right)$



$S' = \{(0,0,1)\}$.

IDEA: Given $S = \{x \in \mathbb{R}^s \mid Ax = b,\ x \geq 0\}$ with $b \geq 0$ in $\mathbb{R}^m$.

Create new polyhedron $S' \subseteq \mathbb{R}^{s+m}$ such that $S' \neq \emptyset$ is guaranteed, by introducing artificial variables $y_1, y_2, \ldots, y_m \geq 0$

Rewrite equation i of $Ax = b$:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{is}x_s + y_i = b_i$$

Matrix form:

$$S' = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{s+m} \;\middle|\; A'\begin{pmatrix} x \\ y \end{pmatrix} = b,\ x \geq 0,\ y \geq 0 \right\}$$

$$A' = \left.\begin{bmatrix} \underbrace{\qquad A \qquad}_{s} & \underbrace{\;I_m\;}_{m} \end{bmatrix}\right\} m$$

Note: S' contains obvious BFS: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \in \mathbb{R}^{s+m}$.

Note: $S \neq \emptyset \iff S'$ contains point of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, with $y = 0$.

(In fact: $x \in S \iff \begin{pmatrix} x \\ 0 \end{pmatrix} \in S'$.)

CLEVER TRICK: Minimize $z' = y_1 + y_2 + \cdots + y_m$

subject to constraints $A'\begin{pmatrix} x \\ y \end{pmatrix} = b$, $x \geq 0$, $y \geq 0$.

If optimal solution satisfies $z' = 0$, then it has form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ where ~~[illegible]~~ $x \in S$ EXTREME POINT.

If optimal solution satisfies $z' > 0$, then $S = \emptyset$.

Example Find extreme point of

$$S = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 = 2 \\ x_1 + x_2 + 4x_3 = 4 \\ x_1, x_2, x_3 \geq 0 \end{array} \right\}$$

Maximize $-z' = -y_1 - y_2$ subject to

$$x_1 + x_2 + x_3 + y_1 = 2$$
$$x_1 + x_2 + 4x_3 + y_2 = 4$$

	x_1	x_2	x_3	y_1	y_2	
y_1	1	1	1	1	0	2
y_2	1	1	4	0	1	4
	0	0	0	1	1	0

BFS : $(0,0,0,2,4) = (x_1, x_2, x_3, y_1, y_2)$

Small problem : Must have zeros in objective row under basic variables.
Fix with row operations!

	x_1	x_2	$\downarrow x_3$	y_1	y_2	
y_1	1	1	1	1	0	2
$\leftarrow y_2$	1	1	$\textcircled{4}$	0	1	4
	-2	-2	-5	0	0	-6

$\theta_1 = 2$ Enter : x_3

$\theta_2 = 1$ Depart : y_2

	$\swarrow x_1$	x_2	x_3	y_1	y_2	
$\leftarrow y_1$	$\textcircled{\frac{3}{4}}$	$\frac{3}{4}$	0	1	$-\frac{1}{4}$	1
x_3	$\frac{1}{4}$	$\frac{1}{4}$	1	0	$\frac{1}{4}$	1
	$-\frac{3}{4}$	$-\frac{3}{4}$	0	0	$\frac{5}{4}$	-1

$\theta_1 = \frac{4}{3}$ Enter : x_1

$\theta_2 = 4$ Depart : y_1

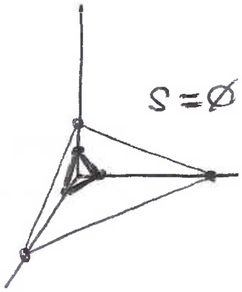
	x_1	x_2	x_3	y_1	y_2	
x_1	1	1	0	$\frac{4}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$
x_3	0	0	$+1$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	0	0	0	1	1	0

BFS : $(\frac{4}{3}, 0, \frac{2}{3}, 0, 0)$

$\therefore (\frac{4}{3}, 0, \frac{2}{3}) \in S$ EXTREME POINT !!!

<u>Example</u> Find extreme point of

$$S'' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 = \frac{1}{2} \\ x_1 + x_2 + 4x_3 = 4 \\ x_1, x_2, x_3 \ge 0 \end{array} \right\}$$



Maximize $-z' = -y_1 - y_2$ subject to

$$x_1 + x_2 + x_3 + y_1 = \frac{1}{2}$$
$$x_1 + x_2 + 4x_3 + y_2 = 4$$

	x_1	x_2	x_3	y_1	y_2	
y_1	1	1	1	1	0	$\frac{1}{2}$
y_2	1	1	4	0	1	4
	0	0	0	1	1	0

BFS: $(0, 0, 0, \frac{1}{2}, 4)$

	x_1	x_2	$\downarrow$ x_3	y_1	y_2	
$\leftarrow y_1$	1	1	①	1	0	$\frac{1}{2}$
y_2	1	1	4	0	1	4
	-2	-2	-5	0	0	$-9/2$

$\theta_1 = 1/2$

$\theta_2 = 1$

Enter: x_3

Depart: y_1

	x_1	x_2	x_3	y_1	y_2	
x_3	1	1	1	1	0	$\frac{1}{2}$
y_2	-3	-3	0	-4	1	2
	3	3	0	5	0	-2

<u>optimal</u> BFS:
$(0, 0, \frac{1}{2}, 0, 2)$

$$\therefore \; S'' = \emptyset \subseteq \mathbb{R}^3.$$

Linear Opt. 10/13/2015

Two-Phase Algo. for Canonical Problem

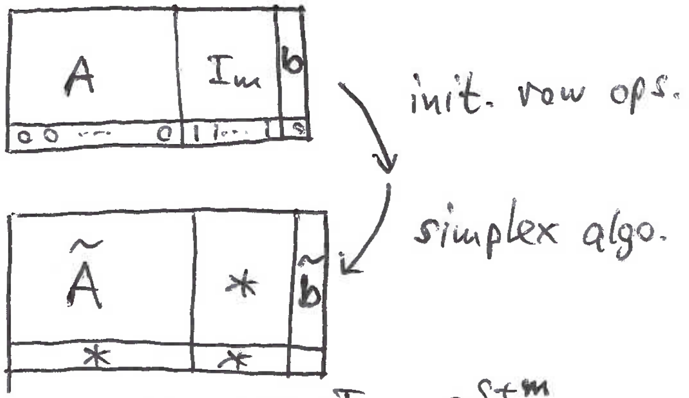
Maximize $z = c^T x$ subject to $Ax = b$, $x \geq 0$. A $m \times s$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^s$.

Phase 1: Find initial BFS to $Ax = b$, $x \geq 0$.

Maximize $-z' = -y_1 - y_2 - \cdots - y_m$ subject to

$$A' \begin{bmatrix} x \\ y \end{bmatrix} = b, \quad x \geq 0, \ y \geq 0.$$

$$A' = \begin{bmatrix} A & I_m \end{bmatrix}$$



Optimal BFS : $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_s, \tilde{y}_1, \ldots, \tilde{y}_m)^T \in \mathbb{R}^{s+m}$

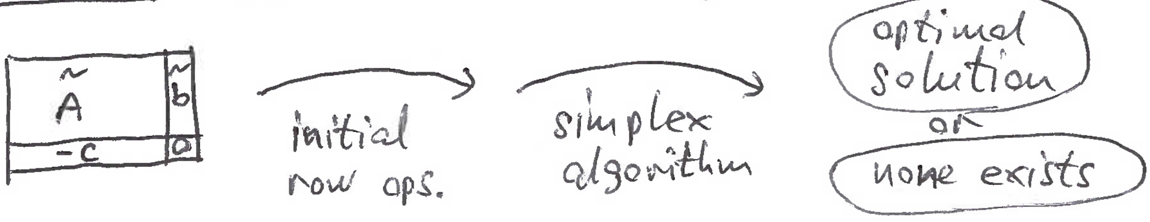
If $(\tilde{y}_1, \ldots, \tilde{y}_m) \neq (0, 0, \ldots, 0)$: Original problem has no feasible solutions. STOP.

Assume $(\tilde{y}_1, \ldots, \tilde{y}_m) = (0, 0, \ldots, 0)$.

OBS : (1) $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_s)$ is obvious BFS to $\tilde{A}x = \tilde{b}$, $x \geq 0$

(2) $Ax = b \Leftrightarrow \tilde{A}x = \tilde{b}$ — Equivalent constraints!!

Phase 2: Maximize $z = c^T x$ subject to $\tilde{A}x = \tilde{b}$, $x \geq 0$



Example

Maximize $z = x_1 + x_2 + x_3$

subj. to $3x_1 + x_2 + 3x_3 = 9$

$3x_1 - x_2 = 3$

$x \geq 0$

$$\left[\begin{array}{ccccc|c} 3 & 1 & 3 & 1 & 0 & 9 \\ 3 & -1 & 0 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r|ccccc|c} & \downarrow x_1 & & & & & \\ \hline y_1 & 3 & 1 & 3 & 1 & 0 & 9 \\ \leftarrow y_2 & \textcircled{3} & -1 & 0 & 0 & 1 & 3 \\ \hline & -6 & 0 & -3 & 0 & 0 & -12 \end{array}$$

$$\begin{array}{r|ccccc|c} & & \cancel{x_2} \downarrow & \downarrow x_3 & & & \\ \hline \leftarrow y_1 & 0 & 2 & \textcircled{3} & 1 & -1 & 6 \\ x_1 & 1 & -1/3 & 0 & 0 & 1/3 & 1 \\ \hline & 0 & -2 & -3 & 0 & 2 & -6 \end{array}$$

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & x_3 & y_1 & y_2 & \\ \hline x_3 & 0 & 2/3 & 1 & 1/3 & -1/3 & 2 \\ x_1 & 1 & -1/3 & 0 & 0 & 1/3 & 1 \\ \hline & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

BFS =

$\bar{x} = (1, 0, 2, 0, 0)$

Tableau :

$$\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
x_3 & 0 & \frac{2}{3} & 1 & 2 \\
x_1 & 1 & -\frac{1}{3} & 0 & 1 \\
\hline
 & -1 & -1 & -1 & 0 \\
\hline
\end{array}$$

$$\begin{array}{c|ccc|c}
 & x_1 & \overset{\downarrow}{x_2} & x_3 & \\
\hline
\leftarrow x_3 & 0 & \boxed{\frac{2}{3}} & 1 & 2 \\
x_1 & 1 & -\frac{1}{3} & 0 & 1 \\
\hline
 & 0 & -\frac{2}{3} & 0 & 3 \\
\hline
\end{array}$$

$$\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
x_2 & 0 & 1 & \frac{3}{2} & 3 \\
x_1 & 1 & 0 & \frac{1}{2} & 2 \\
\hline
 & 0 & 0 & 1 & 5 \\
\hline
\end{array}$$

optimal solution : $x = (2, 3, 0)^T$

$z(x) = 5.$

Linear Opt. 10/16/2015

Duality

Factory produces n products $P_1, P_2, \ldots, P_n$ and uses m resources $R_1, R_2, \ldots, R_m$ to do so.

Producing 1 unit of P_j requires a_{ij} units of R_i, and generates a profit of c_j.

Only b_i units of R_i are available.

Maximize profit by linear programming problem:

x_j = # units of P_j produced.

Maximize $z = c^T x$ subject to $Ax \leq b$, $x \geq 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Let $\tilde{x} \in \mathbb{R}^n$ be optimal solution.

$z(\tilde{x}) = c^T \tilde{x}$ optimal profit. $z(\tilde{x}) = c_1\tilde{x}_1 + c_2\tilde{x}_2 + \cdots + c_n\tilde{x}_n$

Q: How to increase profit?

(1) Change $c \in \mathbb{R}^n$.

Easy to understand: If we increase c_j by 1, will increase $z(\tilde{x})$ by (at least) $\tilde{x}_j$.

BUT: Might be difficult to do.

(2) Change A. E.g. R_i = work done by employee # i.

a_{ij} = # hours required by employee # i to produce 1 unit of P_j.

Possible solution : Tell employee #i to work faster
and invest in good whip!

Will assume matrix A represents what is physically possible.

(3) Change $b \in \mathbb{R}^m$.
Relatively easy to do (pay for overtime, hire more people, build new factory, buy new machine, etc.)

Goal: Understand how a small change to b affects bottom line.

Assume that we replace b_i with $b_i + \varepsilon_i$ for $1 \leq i \leq m$.
Expect total profit changes by $w_1 \varepsilon_1 + w_2 \varepsilon_2 + \cdots + w_m \varepsilon_m$
for some constants $w_1, w_2, \ldots, w_m \in \mathbb{R}$.

w_i = "marginal value" of i-th resource R_i.

Want: $w \equiv \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \in \mathbb{R}^m$.

Example: Assume optimal solution $\tilde{x} \in \mathbb{R}^n$ satisfies strict inequality :

$$a_{i1}\tilde{x}_1 + a_{i2}\tilde{x}_2 + \cdots + a_{in}\tilde{x}_n < b_i$$

Then NOTHING is gained by increasing b_i.
So $w_i = 0$.

Naive/incorrect calculation with "~~almost~~" correct result:

Assume we increase b_i to $b_i + \varepsilon_i$ for $1 \le i \le m$ and use all extra resources to produce more of P_j (one product).

Inequality i: $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$

Naive guess: Can use extra amount of R_i to increase x_j by $\frac{\varepsilon_i}{a_{ij}}$ which increases profit by $c_j \frac{\varepsilon_i}{a_{ij}}$.

~~Total~~ Average increase in profit from increases to all resources:

$$\frac{1}{m} c_j \left(\frac{\varepsilon_1}{a_{1j}} + \frac{\varepsilon_2}{a_{2j}} + \cdots + \frac{\varepsilon_m}{a_{mj}} \right)$$

So $w \in \mathbb{R}^m$ must satisfy:

$$\frac{1}{m} c_j \left(\frac{\varepsilon_1}{a_{1j}} + \cdots + \frac{\varepsilon_m}{a_{mj}} \right) \le w_1 \varepsilon_1 + w_2 \varepsilon_2 + \cdots + w_m \varepsilon_m .$$

Set $\varepsilon_i = a_{ij}$ for $1 \le i \le m$.

$$c_j \le w_1 a_{1j} + w_2 a_{2j} + \cdots + w_m a_{mj} .$$

~~Correct inequality: $a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m \ge c_j$.~~

All ~~correct~~ inequalities, for all j:

$$A^T w \ge c \quad \text{in } \mathbb{R}^n .$$

Duality Theorem: $w \in \mathbb{R}^m$ is the optimal solution to the Dual Problem:

$$\text{Minimize} \quad z' = b^T w \quad \text{subject to} \quad A^T w \ge c, \quad w \ge 0 .$$

Primal Problem

34 31

Maximize $z =$ ~~[illegible]~~ $x +$ ~~[illegible]~~ y

subject to $5x + 2y \le 16$

$3x + 7y \le 27$

$x, y \ge 0.$

$(x,y) = (2,3$

$z =$ ~~[illegible]~~

161

Dual Problem:

Minimize $z' = 16u + 27v$

subject to $5u + 3v \ge$ ~~[illegible]~~ 34

$2u + 7v \ge$ ~~[illegible]~~ 31

$u, v \ge 0$

$(u,v) = (5,3)$

$z' = 161$

Maximize $z = 3x + 5y$

subj. to $x + 2y \leq 10$

$x, y \geq 0$

$(x, y) = (10, 0)$

$z = 30 =$ ~~3x + [illegible]~~

$3 \cdot b$

Minimize $z' = 10w$

subj. to $w \geq 3$

$2w \geq 5$

$w \geq 0$

$w = 3$

$z' = 30$

Maximize $z = 7w$

subject to $3w \leq 2$

$5w \leq 6$

$w \geq 0$

$w = \frac{2}{3}$

$z(w) = \frac{14}{3} = \frac{7}{3} b_1$

Minimize $z' = 2x + 6y$

subject to $3x + 5y \geq 7$

$x, y \geq 0$

$(x, y) = (\frac{7}{3}, 0)$

$z' = \frac{14}{3}$

Duality.

Maximize $z = c^T x$
subject to $Ax \le b$, $x \ge 0$.

A $m \times n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$
b_i = availability of resource R_i
x_j = amount produced of product P_j

Let $\tilde{x} \in \mathbb{R}^n$ be optimal solution.
$z(\tilde{x})$ = optimal profit.

Q: How does optimal profit depend on b?

Assume that we replace b_i with $b_i + \varepsilon_i$ for $1 \le i \le m$.
Expect total profit increases by $w_1\varepsilon_1 + w_2\varepsilon_2 + \cdots + w_m\varepsilon_m$
for some constants $w_1, w_2, \ldots, w_m \in \mathbb{R}$.

Want: $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \in \mathbb{R}^m$.

Incorrect calculation with correct result:

Assume we increase b_i to $b_i + \varepsilon_i$ for $1 \le i \le m$
and use all extra resources to increase on x_j
(more of one product P_j).

Inequality i: $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$

Naive guess: Can use extra amount of resource R_i
to increase x_j by $\dfrac{\varepsilon_i}{a_{ij}}$
which increases profit by $c_j \dfrac{\varepsilon_i}{a_{ij}}$.

Average increase in profit from increases to all resources:

$$\frac{1}{m} c_j \left(\frac{\varepsilon_1}{a_{1j}} + \frac{\varepsilon_2}{a_{2j}} + \cdots + \frac{\varepsilon_m}{a_{mj}} \right)$$

So $w \in \mathbb{R}^m$ must satisfy:

$$\frac{1}{m} c_j \left(\frac{\varepsilon_1}{a_{1j}} + \cdots + \frac{\varepsilon_m}{a_{mj}} \right) \leq w_1 \varepsilon_1 + w_2 \varepsilon_2 + \cdots + w_m \varepsilon_m .$$

Set $\varepsilon_i = a_{ij}$ for $1 \leq i \leq m$.

$$c_j \leq w_1 a_{1j} + w_2 a_{2j} + \cdots + w_m a_{mj} .$$

All inequalities, for all j:

$$A^T w \geq c \quad \text{in } \mathbb{R}^n.$$

Duality Theorem: $w \in \mathbb{R}^m$ is the optimal solution to

Dual Problem:

Minimize $z' = b^T w$ subject to $A^T w \geq c$, $w \geq 0$

Example

Primal Problem:

Maximize $z = 34x_1 + 31x_2$

subject to
$$5x_1 + 2x_2 \leq 16$$
$$3x_1 + 7x_2 \leq 27$$
$$x_1, x_2 \geq 0$$

Optimal solution: $\tilde{x} = (2,3)$

$$z(\tilde{x}) = c^T \tilde{x} = 34 \cdot 2 + 31 \cdot 3 = 161$$

Dual Problem:

Minimize $z' = 16w_1 + 27w_2$

subject to
$$5w_1 + 3w_2 \geq 34$$
$$2w_1 + 7w_2 \geq 31$$
$$w_1, w_2 \geq 0$$

optimal solution:

$\tilde{w} = (5,3)$

$z'(\tilde{w}) = b^T \tilde{w}$

$= 16 \cdot 5 + 27 \cdot 3 = 161$

∴ If we increase b_i by 1, then total profit increases by $\tilde{w}_i$.

Interpretation of $\tilde{w} \in \mathbb{R}^m$

Maximize $z = c^T x$
subject to $Ax \leq b$, $x \geq 0$

Duality Theorem $\Rightarrow \exists\ \tilde{w} \in \mathbb{R}^m$ such that

$$\text{optimal profit} = b^T \tilde{w} = b_1 \tilde{w}_1 + b_2 \tilde{w}_2 + \dots + b_m \tilde{w}_m$$

$\tilde{w}$ tells us what all resources are worth to **bottom** line.
NOT market values!

$\tilde{w}_i$ = <u>marginal value</u> of resource R_i
= amount added/subtracted to ~~[illegible]~~ total profit if one unit of R_i is added/removed.

Constraint # i : $a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \leq b_i$

Producing one unit of P_j requires a_{ij} units of R_i for $1 \leq i \leq m$.

One unit of P_j gives profit c_j.

<u>Dual inequality</u>:

$$a_{1j} \tilde{w}_1 + a_{2j} \tilde{w}_2 + \dots + a_{mj} \tilde{w}_m \geq c_j$$

Says: The resources required to produce one unit of P_j
are worth ~~at least~~ (to us, bottom line!)
at least as much as the profit ~~gained~~
generated by one unit of P_j.

<u>Dual Problem</u>: Minimize $z' = b^T w$ subj. to $Aw \geq c$, $w \geq 0$.

Minimize the total value (to us) of all resources.

Formal Treatment.

Def Dual of problem in Std. Form. **Primal vs Dual**

Thm Dual of Dual prob is primal problem.

Thm Dual of Canonical Problem:

Max $z = c^T x$ subj to $Ax = b$, $x \geq 0$

is

Min $z' = b^T w$ subj to $A^T w \geq c$

w unrestricted.

Thm Dual of

Maximize $z = c^T x$

subj to $Ax \leq b$, x unres.

is

Minimize $z' = b^T w$

subj to $A^T w = c$, $w \geq 0$.

General translation.

Primal Problem

Maximize $z = c^T x$

subject to $Ax \leq b$, $x \geq 0$

Dual Problem

Minimize $z' = b^T w$

subject to $A^T w \geq c$, $w \geq 0$.

A $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Variables: $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$.

Last time: Dual of dual problem is primal problem.

Dual of Canonical Problem:

Thm The dual of

Maximize $z = c^T x$

subj. to $Ax = b$, $x \geq 0$

is

Minimize $z' = b^T w$

subj. to $A^T w \geq 0$, w unrest.

Proof

Primal problem in Std. form:

Maximize $z = c^T x$

subject to

$$\left.\begin{aligned} Ax &\leq b \\ -Ax &\leq -b \end{aligned}\right\} \quad \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}$$

$$x \geq 0$$

Min $z' = \begin{bmatrix} b^T & -b^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

$\begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq c$

Dual Problem: Minimize $z' = b^T u - b^T v$, $u, v \in \mathbb{R}^m$

subject to $A^T u - Av \geq c$, $u, v \geq 0$.

Set $w = u - v \in \mathbb{R}^m$.

Equivalent to: Minimize $z' = b^T w$

subject to $A^T w \geq c$,

$w = u - v$ unrestricted.

$\square$

Example

Primal: Maximize $z = x_1 + 2x_2 - x_3$

Subj. to
$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

Feasible solutions $\subseteq \mathbb{R}^3$: line segment between $(0,1,5)$ and $(\frac{5}{2}, \frac{7}{2}, 0)$

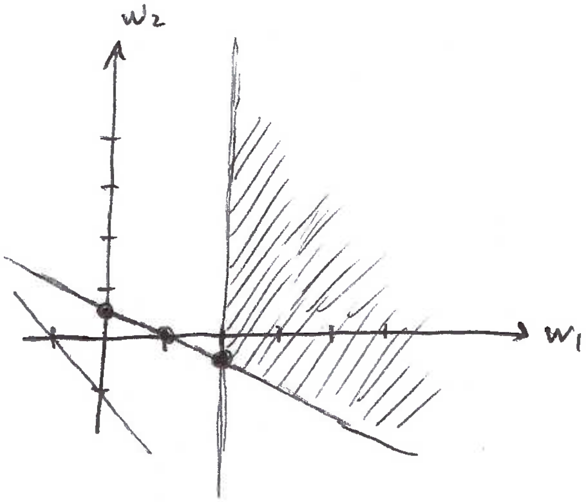
Optimal sol: $\tilde{x} = (\frac{5}{2}, \frac{7}{2}, 0)$, $z(\tilde{x}) = \frac{19}{2}$

Dual: Minimize $z' = 6w_1 + 5w_2$

Subject to
$$w_1 + 2w_2 \geq 1$$
$$w_1 \geq 2$$
$$w_1 + w_2 \geq -1$$

Optimal sol: $\tilde{w} = (2, -\frac{1}{2})$

$z'(\tilde{w}) = \frac{19}{2}$.



Thm The dual of

Maximize $z = c^T x$

subj. to $Ax \leq b$,

x unrestricted.

is:

Minimize $z' = b'w$

Subj. to $A^T w = c$,

$w \geq 0$

Proof

Set $x = u - v$, $u, v \geq 0$ in $\mathbb{R}^n$.

Primal problem in ~~canonical~~ std. form:

Maximize $z = [c^T \ -c^T] \begin{bmatrix} u \\ v \end{bmatrix}$

Subject to

$$\begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leq b, \quad u, v \geq 0$$

Dual Problem :

Minimize $\boxed{z' = b^T w}$ subject to

$$\begin{bmatrix} A^T \\ -A^T \end{bmatrix} w \geq \begin{bmatrix} c \\ -c \end{bmatrix}, \quad w \geq 0$$

Note : $A^T w \geq c$ and $-A^T w \geq -c \Leftrightarrow A^T w = c$.

□

General Procedure for dualizing

Primal Problem : Maximize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to

$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$ ~~OR~~ $\boxed{\leq b_i}$ OR $\boxed{= b_i}$

for $1 \leq i \leq m$

And $\boxed{x_j \geq 0}$ OR $\boxed{x_j \text{ unrestricted}}$, $1 \leq j \leq n$.

Dual Problem : Minimize $z' = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$

subject to

$a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m$ $\geq c_j$ $\quad = c_j$

for $1 \leq j \leq n$

$\uparrow$

if $\boxed{x_j \geq 0}$ | if $\boxed{x_j \text{ unrest}}$

And $w_i \geq 0$ if $\boxed{\leq b_i}$

w_i unrest. if $\boxed{= b_i}$

Compact form:

Maximize $z = c^T x$	$\xleftrightarrow{\text{dualize}}$	Minimize $z' = b^T w$
$\leq b_i$	$\longleftrightarrow$	$w_i \geq 0$
$= b_i$	$\longleftrightarrow$	w_i unrest.
$x_j \geq 0$	$\longleftrightarrow$	$\geq c_j$
x_j unrest.	$\longleftrightarrow$	$= c_j$

Note We can go both ways!

Example: Find dual of:

Minimize $z' = 3x_2 + x_3$

subject to

$$x_1 + 3x_2 \leq 10$$

$$2x_1 - x_2 + x_3 \geq 5$$

$$5x_1 - 3x_2 + 4x_3 = 15$$

$$x_1 \geq 0, \quad x_2, x_3 \text{ unrest.}$$

For minimization problem, formulate all constraints as $=$ or $\geq$: (same problem:)

Minimize $z = 3x_2 + x_3$

subject to

$$-x_1 - 3x_2 \geq -10$$

$$2x_1 - x_2 + x_3 \geq 5$$

$$5x_1 - 3x_2 + 4x_3 = 15$$

$$x_1 \geq 0, \quad x_2, x_3 \text{ unrestricted.}$$

Dual Problem:

$$
\begin{aligned}
\text{Maximize} \quad & z' = -10w_1 + 5w_2 + 15w_3 \\
\text{subject to} \quad & -w_1 + 2w_2 + 5w_3 \leq \textcircled{2} \\
& -3w_1 - w_2 - 3w_3 = 3 \\
& w_2 + 4w_3 = 1 \\
& w_1 \geq 0, \quad w_2 \geq 0, \quad w_3 \text{ unrestricted.}
\end{aligned}
$$

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Examples where both Primal and Dual problems have no feasible solutions.

Example 1

Primal:

$$\begin{aligned} &\text{Maximize } 1\cdot x \\ &\text{Subject to } 0\cdot x \le -1 \\ &\qquad x \ge 0 \end{aligned}$$

Dual:

$$\begin{aligned} &\text{Minimize } -1\cdot w \\ &\text{subject to } 0\cdot w \ge 1 \\ &\qquad w \ge 0 \end{aligned}$$

Example 2

Primal:

$$\begin{aligned} &\text{Maximize } x_1 + x_2 \\ &\text{Subject to } x_1 - x_2 \le -1 \\ &\qquad -x_1 + x_2 \le -1 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}$$

Dual:

$$\begin{aligned} &\text{Minimize } -w_1 - w_2 \\ &\text{Subject to } w_1 - w_2 \ge 1 \\ &\qquad -w_1 + w_2 \ge 1 \\ &\qquad w_1, w_2 \ge 0 \end{aligned}$$

Weak Duality Theorem

Primal:

$$\begin{aligned} &\text{Maximize } c^T x \\ &\text{Subject to } Ax \le b \\ &\qquad x \ge 0 \text{ in } \mathbb{R}^n \end{aligned}$$

Dual:

$$\begin{aligned} &\text{Minimize } b^T w \\ &\text{Subject to } A^T w \ge c \\ &\qquad w \ge 0 \text{ in } \mathbb{R}^m \end{aligned}$$

A $m \times n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Thm

Let $\tilde{x}$ be any feasible solution to primal problem, and let $\tilde{w}$ be any feasible solution to dual problem.

Then $c^T \tilde{x} \le b^T \tilde{w}$.

Proof

Since $\tilde{x}$ feasible: $A\tilde{x} \le b$. Since $\tilde{w} \ge 0$: $\tilde{w}^T A \tilde{x} \le \tilde{w}^T b$.

Since $\tilde{w}$ feasible: $A^T \tilde{w} \ge c$. Equivalent: $c^T \le \tilde{w}^T A$.

Since $\tilde{x} \ge 0$: $c^T \tilde{x} \le \tilde{w}^T A \tilde{x}$

$\therefore$ $c^T \tilde{x} \le \tilde{w}^T A \tilde{x} \le \tilde{w}^T b = b^T \tilde{w}$.

$\square$

Cor If $c^T \tilde{x} = b^T \tilde{w}$ then $\tilde{x}$ and $\tilde{w}$ are both optimal solutions.

Cor (a) If primal problem has feasible solutions but objective function $z = c^T x$ is not bounded above, then dual problem has no feasible solutions.

(b) If dual problem has feasible solutions but objective function $z' = b^T w$ is not bounded below, then primal problem has no feasible solutions.

Example

Primal:

Maximize $z = x_1 + 4x_2$

subject to $2x_1 - x_2 \leq 3$

$x_1, x_2 \geq 0$

z unbounded

Dual:

Minimize $3w_1$

subject to $2w_1 \geq 1$

$-w_1 \geq 4$, $\boxed{w_1 \geq 0}$

no feasible solutions.

Consider Canonical Problem:

Maximize $c^T x$

subject to $Ax = b$, $x \geq 0$

A $m \times s$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^s$.

Assume this problem has BFS $\tilde{x} \in \mathbb{R}^n$ with basic variables $x_{k_1}, x_{k_2}, \dots, x_{k_m}$.

Q: How to find tableau of this BFS?

Start with

$$\begin{array}{|c|c|} \hline A & b \\ \hline -c^T & 0 \\ \hline \end{array}$$

then create pivot at entry (i, k_i) for $1 \leq i \leq m$.

Write $A = \begin{array}{|c|c|c|c|} \hline A_1 & A_2 & \cdots & A_s \\ \hline \end{array}$, A_j = column #j in A.

Notes lost

$$B = \left[\begin{array}{c|c|c|c} A_{k_1} & A_{k_2} & \cdots & A_{k_m} \end{array}\right] \quad m \times m.$$

$$\tilde{x}_B = \begin{bmatrix} \tilde{x}_{k_1} \\ \vdots \\ \tilde{x}_{k_m} \end{bmatrix} = B^{-1} b$$

Show: $B^{-1} A_{k_i} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow i$

$$c_B = \begin{bmatrix} c_{k_1} \\ \vdots \\ c_{k_m} \end{bmatrix}$$

Tableau representing $\tilde{x}$:

$$\begin{array}{c|c|c|} x_{k_1} & & \\ \vdots & B^{-1}A & B^{-1}b \\ x_{k_m} & & \\ \hline & c_B^T B^{-1} A - c^T & \cdot \end{array} \qquad \circledast$$

$c_B^T B^{-1} b$

Example

Thm Assume the canonical problem has an optimal solution $\tilde{x} \in \mathbb{R}^s$. Then its dual problem also has an optimal solution $\tilde{w} \in \mathbb{R}^m$, and we have $c^T \tilde{x} = b^T \tilde{w}$.

Proof

Let $\tilde{x}$ be represented by tableau $\circledast$. (final tableau) Granting that simplex algo works, this tableau has non-negative objective row.

Set $\tilde{w}^T = c_B^T B^{-1}$.

Check that $b^T \tilde{w} = c^T \tilde{x}$ and $A^T \tilde{w} \geq c$.

□

Primal Problem:

Maximize $c^T x$
subject to $Ax \leq b$, $x \geq 0$ in $\mathbb{R}^n$

A $m \times n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Dual Problem:

Minimize $b^T w$
Subject to $A^T w \geq c$, $w \geq 0$ in $\mathbb{R}^m$

Duality Theorem

The following are equivalent:

(a) Both the primal and dual problems have feasible solutions.

(b) The primal problem has an optimal solution.

(c) The dual problem has an optimal solution.

When this is the case we have $c^T \tilde{x} = b^T \tilde{w}$, where $\tilde{x} \in \mathbb{R}^s$ and $\tilde{w} \in \mathbb{R}^m$ are optimal solutions to the primal and dual problems.

Proof If (a) is true, then $z = c^T x$ is bounded above and $z' = b^T w$ is bounded below (Weak Dual Thm.)
This implies that both problems have optimal solutions.
(Really requires more careful argument.
But follows from fact that simplex algorith works.
Two-phase algorithm will tell us:
No feasible solc, OR objective function unbounded, OR optimal solution)

Enough to prove (b) $\Rightarrow$ (c).

If (b) true, then canonical problem has opt. sol:

Maximize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$
Subject to $\begin{bmatrix} A & I_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix} = b$, $x \geq 0$ in $\mathbb{R}^s$, $s = n+m$.

Thm $\Rightarrow$ Dual problem has opt sol:

Minimize $b^T w$
Subj. to $\begin{bmatrix} A^T \\ I_m \end{bmatrix} w \geq \begin{bmatrix} c \\ 0 \end{bmatrix}$.

Constraints are: $A^T w \geq c$
$w \geq 0$.

□

Recall: Canonical Problem:

Maximize $z = c^T x$ $\quad$ A $m \times s$

subj. to $Ax = b$, $x \geq 0$ in $\mathbb{R}^s$ $\quad$ $b \in \mathbb{R}^m$, $c \in \mathbb{R}^s$.

Assume $\tilde{x}$ is a BFS using basic variables $x_{k_1}, \ldots, x_{k_m}$.

A_j = j-th column from A.

$$B = \left[\begin{array}{c|c|c|c} A_{k_1} & A_{k_2} & \cdots & A_{k_m} \end{array}\right] \quad m \times m.$$

$$c_B = \begin{bmatrix} c_{k_1} \\ \vdots \\ c_{k_m} \end{bmatrix} \qquad \tilde{x}_B = \begin{bmatrix} \tilde{x}_{k_1} \\ \vdots \\ \tilde{x}_{k_m} \end{bmatrix}$$

Tableau representing $\tilde{x}$:

$$\begin{array}{c|c|c|l} \cline{2-3} x_{k_1} & & & \\ \vdots & B^{-1}A & B^{-1}b & \\ x_{k_m} & & & \\ \cline{2-3} & c_B^T B^{-1} A - c^T & \circ & \!\!- c_B^T B^{-1} b \\ \cline{2-3} \end{array}$$

Thm If Canonical problem has opt sol $\tilde{x}$ then ~~so does~~ dual problem has opt sol $\tilde{w}$ and $c^T \tilde{x} = b^T \tilde{w}$.

Proof: Check that $\tilde{w}^T = c_B^T B^{-1}$ is optimal sol to dual problem.

Primal Problem:

Maximize $c^T x$
subject to $Ax \le b$, $x \ge 0$ in $\mathbb{R}^n$

A $m \times n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Dual Problem:

Minimize $b^T w$
Subject to $A^T w \ge c$, $w \ge 0$ in $\mathbb{R}^m$

Duality Theorem

The following are equivalent:

(a) Both the primal and dual problems have feasible solutions.

(b) The primal problem has an optimal solution.

(c) The dual problem has an optimal solution.

When this is the case we have $c^T \tilde{x} = b^T \tilde{w}$, where $\tilde{x} \in \mathbb{R}^s$ and $\tilde{w} \in \mathbb{R}^m$ are optimal solutions to the primal and dual problems.

Proof If (a) is true, then $z = c^T x$ is bounded above and $z' = b^T w$ is bounded below (Weak Dual Thm.)
This implies that both problems have optimal solutions.
(Really requires more careful argument.
But follows from fact that simplex algorith works.
Two-phase algorithm will tell us:
No feasible solc, OR objective function unbounded, OR optimal solution.)

Enough to prove (b) ⇒ (c).

If (b) true, then canonical problem has opt. sol:

Maximize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to $\begin{bmatrix} A & I_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix} = b$, $x \ge 0$ in $\mathbb{R}^s$, $s = n+m$.

Thm ⇒ Dual problem has opt sol:

Minimize $b^T w$

Subj. to $\begin{bmatrix} A^T \\ I_m \end{bmatrix} w \ge \begin{bmatrix} c \\ 0 \end{bmatrix}$.

Constraints are: $A^T w \ge c$
$w \ge 0$.

□

Linear Opt. 10/30/2015 ①

Shortcut: Solve primal and dual problems at the same time.

Assume we have problem in Std. form with $b \geq 0$:

Maximize $z = c^T x$
subj. to $Ax \leq b$, $x \geq 0$. in $\mathbb{R}^n$

A $m \times n$
$b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $\boxed{b \geq 0}$

Translate to canonical problem:

Maximize $z = c_1 x_1 + \cdots + c_n x_n$
subj. to $A'x = b$, $x \geq 0$ in $\mathbb{R}^s$.

$s = n + m$

$$A' = \begin{array}{|c|c|} \hline \quad A \quad & I_m \\ \hline \end{array}$$

Initial tableau:

$$\begin{array}{c|c|c|c|} \hline x_{n+1} & & & \\ \vdots & A & I_m & b \\ x_s & & & \\ \hline & -c^T & 0 & \\ \hline \end{array}$$

Final tableau:

$$\begin{array}{c|c|c|c|} \hline x_{k_1} & & & \\ \vdots & B^{-1}A & B^{-1}I_m & B^{-1}b \\ x_{k_m} & & & \\ \hline & c_B^T B^{-1} A - c^T & c_B^T B^{-1} & \xleftarrow{} c_B^T B^{-1} b \\ \hline \end{array}$$

B = $m \times m$ matrix of columns of final basic variables.

in initial tableau

$$c_B = \begin{bmatrix} c_{k_1} \\ \vdots \\ c_{k_m} \end{bmatrix}$$ where $c_i = 0$ for $i > n$.

Note: $\tilde{w}^T = c_B^T B^{-1} =$ [illegible] last m entries in obj. row of final tableau!

$\tilde{x}_B = B^{-1} b$ = last column.

Example

Max $x_1 + x_2$

Subj. to $2x_1 + 3x_2 \le 16$

$3x_1 + 2x_2 \le 19$

$5x_1 + x_2 \le 30$

		↓ x_2				
← x_3	2	③	1	0	0	16
x_4	3	2	0	1	0	19
x_5	5	1	0	0	1	30
	-1	-1	0	0	0	0

$\theta_1 = \frac{16}{3}$

$\theta_2 = \frac{19}{2}$

$\theta_3 = 30$

	↓ x_1					
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	$\frac{16}{3}$
← x_4	$\frac{5}{3}$	0	$-\frac{2}{3}$	1	0	$\frac{25}{3}$
x_5	$\frac{13}{3}$	0	$-\frac{1}{3}$	0	1	$\frac{74}{3}$
	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{16}{3}$

$\theta_1 = 8$

$\theta_2 = 5$

$\theta_3 = \frac{74}{13}$

x_2	0	1	$\frac{3}{5}$	$-\frac{2}{5}$	0	2
x_1	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	0	5
x_5	0	0	$\frac{7}{5}$	$-\frac{13}{5}$	1	3
	0	0	$\frac{1}{5}$	$\frac{1}{5}$	0	7

Find B, B^{-1},

$\tilde{x}_B$, $\tilde{x}$, $\tilde{w}$.

$z(\tilde{x})$.

Dual simplex Algo.

Example :

Minimize $z = x_1 + 2x_2$

subj to $x_1 - 2x_2 + x_3 \geq 4$

$2x_1 + x_2 - x_3 \geq 6$

$x \geq 0$.

Max $-z = -x_1 - 2x_2$

$-x_1 + 2x_2 - x_3 + x_4 = -4$

$-2x_1 - x_2 + x_3 + x_5 = -6$

$x_j \geq 0 \quad , \quad 1 \leq j \leq 5$.

x_4	-1	2	-1	1	0	-4
x_5	-2	-1	1	0	1	-6
	1	2	0	0	0	0

Means what?

Dual Simplex Algorithm

Recall: The tableau

$$\begin{array}{|c|c|} \hline A & b \\ \hline -c^T & d \\ \hline \end{array}$$

A $m \times s$
$b \in \mathbb{R}^m$
$c \in \mathbb{R}^s$
$d \in \mathbb{R}$

encodes the linear problem:

Maximize $z = c^T x + d$
Subject to $Ax = b$, $x \geq 0$ in $\mathbb{R}^s$.

- If A contains a pivot in each row, [redacted] and $b \geq 0$, then tableau also encodes an "obvious BFS". It may not be optimal; simplex algorithm will improve it.

- Assume A contains a pivot in each row, and $-c \geq 0$. Then tableau encodes an "obvious basic solution". It may not be feasible, but it is optimal (in a certain sense.)

Example

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline x_1 & 1 & 3 & 0 & 0 & 1 & 12 \\ x_4 & 0 & -8 & 0 & 1 & -3 & -24 \\ x_3 & 0 & -2 & 1 & 0 & -1 & -7 \\ \hline & 0 & 5 & 0 & 0 & 2 & 5 \end{array}$$

Represents:

Maximize $z = 5 - 5x_2 - 2x_5$
Subject to

$$\begin{aligned} x_1 + 3x_2 + x_5 &= 12 \\ -8x_2 + x_4 - 3x_5 &= -24 \\ -2x_2 + x_3 - x_5 &= -7 \\ x \geq 0 \text{ in } \mathbb{R}^5 \end{aligned}$$

Current BS:

$\tilde{x} = (12, 0, -7, -24, 0)$

Note $\tilde{x}$ is optimal solution to:

$$\text{Maximize } z = 5 - 5x_2 - 2x_5$$
$$\text{subject to } Ax = b, \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq -7,\ x_4 \geq -24,\ x_5 \geq 0$$

Goal: Find adjacent basic solution that is also optimal (obj. row ≥ 0) and is closer to being feasible.

	x_1	x_2	x_3	x_4	$\downarrow x_5$	
x_1	1	3	0	0	1	12
x_4	0	-8	0	1	-3	-24
$\leftarrow x_3$	0	-2	1	0	$\boxed{-1}$	-7
	0	5	0	0	2	5

Departing variable: x_3
(could also choose x_4.)

Must add multiple of row 3 to objective row:

Entering variable:

x_2 : add $\frac{5}{2}$ times row 3 to obj.

$\boxed{x_5}$: add 2 times row 3 to obj.

	x_1	$\downarrow x_2$	x_3	x_4	x_5	
x_1	1	1	1	0	0	5
$\leftarrow x_4$	0	$\boxed{-2}$	-3	1	0	-3
x_5	0	2	-1	0	1	7
	0	1	2	0	0	-9

Departing variable: x_4

Entering variable:

$\boxed{x_2}$: add $\frac{1}{2}$ times row 2 to obj.

x_3 : add $\frac{2}{3}$ times row 2 to obj.

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{7}{2}$
x_2	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_5	0	0	-4	1	1	4
	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{21}{2}$

Optimal Solution: $x = \left(\frac{7}{2}, \frac{3}{2}, 0, 0, 4\right)$.

Example

	x_1	$\overset{\downarrow}{x_2}$	x_3	x_4	x_5	x_6	x_7	
← x_4	2	−1	0	1	0	0	−3	−3
x_3	1	3	1	0	7	0	5	25
x_6	4	0	0	0	1	1	−2	10
	1	2	0	0	3	0	10	70

Departing var: x_4

Entering variable :

x_1 : No good.

$\boxed{x_2}$: add 2 × row 1 to obj row.

x_5 : No good.

x_7 : add $\frac{10}{3}$ × row 1 to obj row.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_2	−2	1	0	−1	0	0	3	3
x_3	7	0	1	3	7	0	−4	16
x_6	4	0	0	0	1	1	−2	10
	5	0	0	2	3	0	4	64

Dual Simplex Algorithm

Given tableau $\begin{array}{|c|c|}\hline A & b \\ \hline -c^T & d \\ \hline\end{array}$ such that each row of A has pivot, and $-c \geq 0$.

- If $b \geq 0$ then STOP: Current basic solution is feasible.
- Choose departing variable with negative value in current BS. I.e. choose pivotal row i such that $b_i < 0$.

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{is}x_s = b_i < 0$$

- If all entries in row i are ≥ 0, then STOP. No feasible solutions. (i-th equation impossible!)
- For each negative entry $a_{ij} < 0$ in row i, Compute $\dfrac{-c_j}{a_{ij}}$. Pivotal column is column j such that $\left|\dfrac{c_j}{a_{ij}}\right|$ is minimal.
- Do pivot operation at entry (i,j).

Example Maximize $z = -2x_1 + x_2 - 2x_4 + x_5$

subject to

$$\begin{aligned} -2x_1 + x_2 \quad\quad + 3x_4 \quad\quad &= 3 \\ -x_1 + x_2 \quad\quad + x_4 + x_5 &= 4 \\ 4x_1 \quad\quad + x_3 + 2x_4 + x_5 &= 5 \end{aligned}$$

$$x \geq 0 \text{ in } \mathbb{R}^5.$$

Suppose we have already found the optimal solution $\tilde{x} = (0,3,4,0,1)$, with $z(\tilde{x}) = 4$

Now we realize we also need the constraint

$$x_1 + 2x_2 + x_3 + 4x_4 + x_5 \leq 7.$$

$$x_1 + 2x_2 + x_3 + 4x_4 + x_5 \leq 7$$

Find new optimal solution.

Find tableau representing $\tilde{x}$:

Basic variables ~~x_3, x_4, x_5~~ x_2, x_3, x_5.

$$\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
-2 & \textcircled{1} & 0 & 3 & 0 & 3 \\
-1 & 0 & 0 & 1 & \textcircled{1} & 4 \\
4 & 0 & \textcircled{1} & 2 & 1 & 5 \\
\hline
2 & -1 & 0 & 2 & -1 & 0
\end{array}$$

After pivoting:

$$\begin{array}{c|ccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
x_2 & -2 & 1 & 0 & 3 & 0 & 3 \\
x_5 & 1 & 0 & 0 & -2 & 1 & 1 \\
x_3 & 3 & 0 & 1 & 4 & 0 & 4 \\
\hline
 & 1 & 0 & 0 & 3 & 0 & 4
\end{array}$$

<u>Note</u>: This verifies that $\tilde{x} = (0, 3, 4, 0, 1)$ is an optimal solution.

Add constraint: $x_1 + 2x_2 + 1x_3 + 4x_4 + x_5 + x_6 = 7$

(x_6 new slack variable.)

$$\begin{array}{c|cccccc|cl}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \\
\hline
x_2 & -2 & \textcircled{1} & 0 & 3 & 0 & 0 & 3 & \leftarrow \text{ re-pivot!} \\
x_5 & 1 & 0 & 0 & -2 & \textcircled{1} & 0 & 1 & \leftarrow \\
x_3 & 3 & 0 & \textcircled{1} & 4 & 0 & 0 & 4 & \leftarrow \\
x_6 & 1 & 2 & 1 & 4 & 1 & \boxed{\textcircled{1}} & 7 & \leftarrow \text{ new pivot } \\
\hline
 & 1 & 0 & 0 & 3 & 0 & 0 & 4 &
\end{array}$$

$$\begin{array}{cc|cccccc|c}
 & & x_1 & x_2 & x_3 & \overset{\downarrow}{x_4} & x_5 & x_6 & \\
\hline
 & x_2 & -2 & 1 & 0 & 3 & 0 & 0 & 3 \\
 & x_5 & 1 & 0 & 0 & -2 & 1 & 0 & 1 \\
 & x_3 & 3 & 0 & 1 & 4 & 0 & 0 & 4 \\
\leftarrow & x_6 & 1 & 0 & 0 & \textcircled{-4} & 0 & 1 & -4 \\
\hline
 & & 1 & 0 & 0 & 3 & 0 & 0 & 4
\end{array}$$

Departing: x_6

Entering: x_4

$$\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
x_2 & -\frac{5}{4} & 1 & 0 & 0 & 0 & \frac{3}{4} & 0 \\
x_5 & \frac{1}{2} & 0 & 0 & 0 & 1 & -\frac{1}{2} & 3 \\
x_3 & 4 & 0 & 1 & 0 & 0 & 1 & 0 \\
x_4 & -\frac{1}{4} & 0 & 0 & 1 & 0 & -\frac{1}{4} & 1 \\
\hline
 & \frac{7}{4} & 0 & 0 & 0 & 0 & \frac{3}{4} & 1
\end{array}$$

New optimal solution : $\tilde{x} = (0, 0, 0, 1, 3)$

$z(\tilde{x}) = 1.$

Linear Opt 11/6/2015 ①

Midterm 2 Tuesday November 10 in class.

Sections targeted: 2.2 - 3.4 except "Big M Method".

You should also know: 0.1 - 2.1

Suggestions:

1) Read and understand covered sections.
2) Know statements of Theorems & Definitions.
3) Know how to do HW + MT 1 to perfection.
4) Understand what things mean/represent and how to carry out algorithms.

PAGE 1.5 FIRST

Review

Maximize $z =$ ~~$x_1 + 2x_3 + x_6$~~ ~~$3x_2 - x_3 - x_4 - x_5 - x_6$~~ $\boxed{z = 3x_2 - x_3 - x_4 - x_5 - x_6}$

Subj. to

$$x_1 + x_3 + x_5 = 1$$
$$x_2 + x_3 + x_4 = 1$$
$$x_1 + x_2 + x_6 \leq 2$$
$$x_3 + x_4 + x_5 + 2x_6 \geq 1$$
$$x \geq 0 \text{ in } \mathbb{R}^6.$$

Dual Problem:

Minimize $z' = w_1 + w_2 + 2w_3 - w_4$

subject to

$$w_1 + w_3 \geq 0$$
$$w_2 + w_3 \geq 3$$
$$w_1 + w_2 - w_4 \geq -1$$
$$w_2 - w_4 \geq -1$$
$$w_1 - w_4 \geq -1$$
$$w_3 - 2w_4 \geq -1$$

w_1, w_2 unrestricted.

$w_3, w_4 \geq 0$

Recall: The tableau encodes the linear problem

$$\begin{array}{c|c} A & b \\ \hline -c^T & d \end{array}$$

Maximize $z = c^T x + d$
Subject to
$Ax = b \ , \quad x \geq 0$

Example

Maximize $z = x_1 + x_2 + x_3$

Subject to

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 4 \\ x_1 + x_2 &\geq 2 \\ 3x_1 + 2x_2 &\leq 7 \end{aligned}$$

Note: we negate 2nd equation to form dual problem.

Dual Problem:

Minimize $z' = 4w_1 - 2w_2 + 7w_3$

subject to

$$\begin{aligned} 2w_1 - w_2 + 3w_3 &\geq 1 \\ -w_1 - w_2 + 2w_3 &\geq 1 \\ w_1 &\geq 1 \end{aligned}$$

w_1 unrestricted, $(w_2, w_3) \geq 0$

Phase 1 problem: Slack: x_4, x_5 Artificial: y_1

Maximize $-y_1$

Subject to

$$\begin{aligned} 2x_1 - x_2 + x_3 & & &= 4 \\ x_1 + x_2 & -x_4 & + y_1 &= 2 \\ 3x_1 + 2x_2 & + x_5 & &= 7 \end{aligned}$$

$x, y \geq 0$

Initial basic variables: $x_3 ; y_1 , x_5$

do_review() ;

Keep artificial variables in Phase 2:

(1) Ignore coefficients of artificial variables in obj. row when checking optimality.

(2) Never choose an artificial to be entering variable.

(3) Choose artificial variables to be departing variable whenever possible.

Optimal solution: $\tilde{x} = (0, \frac{7}{2}, \frac{15}{2})$,

$z(\tilde{x}) = 11$

Find optima solution $\tilde{w}$ to dual problem:

- Add c^T to objective row of final tableau.
- Get $\hat{w} \in \mathbb{R}^m$ from this sum by choosing columns of initial basic variables (in phase 1.)
- Get $\tilde{w} \in \mathbb{R}^m$ from $\hat{w}$ by changing sings of entry $\hat{w}_i$ whenever constraint #i in original problem had its sign changed. (i.e. "$\geq b_i$").

c^T + final obj row: $(1,1,1,0,0,0) + (4,0,0,0,1,0)$

$= (5,1,1,0,1,0)$

$\hat{w} = (1,0,1)$ coefs of x_3, y_1, x_5

$\tilde{w} = (1, -0, 1)$ optimal sol. to dual problem.

$\uparrow$ changed sign of 2nd constraint "≥ 2".

$z'(\tilde{w}) = 11 = z(\tilde{x})$.

Justification of method:

Start with canonical problem $\begin{array}{|c|c|} \hline A & b \\ \hline -c^T & 0 \\ \hline \end{array}$ Initial basic vars: $x_{j_1}, x_{j_2}, \dots, x_{j_m}$.

Optimal solution $\tilde{x} \in \mathbb{R}^s$.

Basic variables of $\tilde{x}$: $x_{k_1}, x_{k_2}, \dots, x_{k_m}$.

$$B = \begin{array}{|c|c|c|} \hline A_{k_1} & \cdots & A_{k_m} \\ \hline \end{array} \qquad c_B = \begin{bmatrix} c_{k_1} \\ \vdots \\ c_{k_m} \end{bmatrix}$$

Tableau representing $\tilde{x}$: $\begin{array}{|c|c|} \hline B^{-1}A & B^{-1}b \\ \hline c_B^T B^{-1} A - c^T & \cdot \\ \hline \end{array}$ — $c_B^T B^{-1} b$

Optimal sol. to dual problem: $\hat{w}^T = c_B^T B^{-1}$

Note: $\hat{w}^T A = (\text{obj. row}) + c^T$

$$A_{j_i} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i\text{-th entry.} \qquad \Rightarrow \hat{w}_i^T = (\hat{w}^T A)_{j_i}$$

$$= c_{j_i} + j_i\text{-th entry of obj row.}$$

Note: Dual of canonical problem $\neq$ dual of original primal problem.

i-th constraint negated ("$\geq b_i$")

$\Rightarrow$ all coefs of w_i are negated in dual of canonical problem.

So $\tilde{w}_i = -\hat{w}_i$ where $\tilde{w} \in \mathbb{R}^m$ optimal sol. to dual of primal problem.

Add constraint $x_3 \leq 7$ to problem:

Tableau representing $\tilde{x}$:

	x_1	x_2	x_3	x_4	x_5	
x_3	$\frac{7}{2}$	0	1	0	$\frac{1}{2}$	$\frac{15}{2}$
x_2	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	$\frac{7}{2}$
x_4	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	$\frac{3}{2}$
	4	0	0	0	1	11

$$x_3 + x_6 = 7$$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	$\frac{7}{2}$	0	1	0	$\frac{1}{2}$	$\bigcirc$	$\frac{15}{2}$
x_2	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	$\frac{7}{2}$
x_4	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	$\frac{3}{2}$
x_6	0	0	1	0	0	1	7
	4	0	0	0	1	0	11

`do_reviewOa();`

New optimal sol.: $\tilde{x} = (\frac{1}{7}, \frac{23}{7}, 7)$

Solve both Primal and Dual problems.

Phase 1 problem: Slack vars: x_7, x_8

Artificial: y_1, y_2, y_3

Maximize $z'' = -y_1 - y_2 - y_3$

subject to

$$
\begin{aligned}
&x_1 + x_3 + x_5 \quad + y_1 = 1 \\
&x_2 + x_3 + x_4 \quad + y_2 = 1 \\
&x_1 + x_2 + x_6 \quad + x_7 = 2 \\
&x_3 + x_4 + x_5 + 2x_6 - x_8 + y_3 = 1 \\
&x, y \geq 0
\end{aligned}
$$

`do_review1();`

Optimal solution: $\tilde{x} =$ ~~$(1, 1, 0, 0, 0, \frac{1}{2})$~~ $(\frac{2}{3}, 1, 0, 0, \frac{1}{3}, \frac{1}{3})$

Initial basic variables: y_1, y_2, x_7, y_3

$\hat{w} =$ [illegible] $(-\frac{1}{3}, \frac{8}{3}, \frac{1}{3}, -\frac{2}{3})$

We changed sign of last equation to form dual problem.

$\tilde{w} =$ [illegible] solution to dual problem.

$\tilde{w} = (-\frac{1}{3}, \frac{8}{3}, \frac{1}{3}, \frac{2}{3})$

$z(\tilde{x}) = \frac{8}{3} = z'(\tilde{w})$

`do_review1a();`

`do_cycle();`

Bland's method:

(1) Choose entering variable x_j with smallest possible j.

(2) Choose departing variable x_i with smallest possible i.

`do_bland();`

```
wait := proc()
  system("read x");
  NULL;
end:

pivot_step := proc(i,j)
  global tableau, circleset, entervar, departvar;
  enter(j);
  wait();
  departvar := i;
  circle(i,j);
  wait();
  entervar := 0;
  departvar := 0;
  pivot(i,j);
  wait();
  circleset := {};
end:

pivot_set := proc(ps)
  global tableau, circleset, entervar, departvar;
  local cl;
  circleset := ps;
  redraw();
  wait();
  for cl in ps do
    pivot_inner(cl[1], cl[2]);
  od;
  redraw();
  wait();
  circleset := {};
end:

ex_cycle := [
[1,1,1,1,1,0,0,1],
[1/2,-11/2,-5/2,9,0,1,0,0],
[1/2,-3/2,-1/2,1,0,0,1,0],
[-1,7,1,2,0,0,0,0]]:

cycle_step := proc(i,j)
  global circleset;
  circle(i,j);
  wait();
  pivot(i,j);
  wait();
  circleset := {};
end:

do_cycle := proc()
  global circleset;
  setup(ex_cycle);
  wait();
  cycle_step(2,1);
  cycle_step(3,2);
  cycle_step(2,3);
  cycle_step(3,4);
  cycle_step(2,6);
  cycle_step(3,7);
  cycle_step(2,1);
  cycle_step(3,2);
  cycle_step(2,3);
  cycle_step(3,4);
  cycle_step(2,6);
  cycle_step(3,7);
```

```
ex_review0 := [
[2, -1, 1, 0, 0, 0, 4],
[1, 1, 0, -1, 0, 1, 2],
[3, 2, 0, 0, 1, 0, 7],
[0, 0, 0, 0, 0, 1, 0]]:

do_review0 := proc()
  global tableau;
  setup(ex_review0, ["x","y"], [5,1]);
  wait();
  pivot_set({[1,3],[2,6],[3,5]});
  pivot_step(2,1);

  tableau[4] := [-1,-1,-1,0,0,0,0];
  redraw();
  wait();
  pivot_set({[1,3],[2,1],[3,5]});
  pivot_step(2,2);
  pivot_step(3,4);
end:

ex_review0a := [
[7/2, 0, 1, 0, 1/2, 0, 15/2],
[3/2, 1, 0, 0, 1/2, 0, 7/2],
[1/2, 0, 0, 1, 1/2, 0, 3/2],
[0, 0, 1, 0, 0, 1, 7],
[4, 0, 0, 0, 1, 0, 11]]:

do_review0a := proc()
  global tableau, circleset, entervar, departvar;
  setup(ex_review0a);
  wait();
  pivot_set({[1,3],[2,2],[3,4],[4,6]});
  depart(4);
  wait();
  entervar := 1;
  circle(4,1);
  wait();
  entervar := 0;
  departvar := 0;
  pivot(4,1);
  circleset := {};
  wait();
  clear();
end:
```

Interpretation of $\tilde{w}$:

~~Given~~ ~~Consider~~

Linear problem:

Maximize $z = c^T x$

subj. to $Ax \le b$, $x \ge 0$.

~~optimal profit satis~~

Duality Theorem $\Rightarrow \exists\, \tilde{w} \in \mathbb{R}^m$ such that

optimal profit $= b^T \tilde{w}$

$\tilde{w} \in \mathbb{R}^m$ tells what all resources are worth to bottom line.

Marginal values. NOT market values!

Constraint #i: $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$

Producing one unit of P_j requires a_{ij} units of R_i.

for $1 \le i \le m$.

One unit of P_j gives profit c_j.

Dual Inequality:

$$a_{1j}\tilde{w}_1 + a_{2j}\tilde{w}_2 + \cdots + a_{mj}\tilde{w}_m \ge c_j$$

The resources required to produce one unit of P_j are worth (to us!) at least as much as we gain from selling that unit!

Sensitivity Analysis

Problem P: Maximize $z = c^T x$ $\quad$ A $m \times s$

Subject to $Ax \leq b$, $x \geq 0 \in \mathbb{R}^s$ $\quad$ $b \in \mathbb{R}^m$

$c \in \mathbb{R}^s$

Represented by tableau:

$$\begin{array}{|c|c|} \hline A & b \\ \hline -c^T & 0 \\ \hline \end{array}$$

But: This tableau may not encode any basic solution.

Find tableau representing basic solution $\tilde{x} \in \mathbb{R}^s$ using basic variables $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$:

$$B = \begin{bmatrix} | & | & & | \\ A_{k_1} & A_{k_2} & \cdots & A_{k_m} \\ | & | & & | \end{bmatrix} \qquad c_B = \begin{bmatrix} c_{k_1} \\ c_{k_2} \\ \vdots \\ c_{k_m} \end{bmatrix}$$

Set $\sigma = c_B^T B^{-1} A \in \mathbb{R}^s$ column vector.

Tableau encoding $\tilde{x}$:

$$\tilde{x}_B = \begin{bmatrix} \tilde{x}_{k_1} \\ \vdots \\ \tilde{x}_{k_m} \end{bmatrix} = B^{-1} b \in \mathbb{R}^m.$$

$$\begin{array}{c|c|c|} \cline{2-3} x_{k_1} & & \\ \vdots & B^{-1}A & B^{-1}b \\ x_{k_m} & & \\ \cline{2-3} & \sigma - c^T & \cdot \\ \cline{2-3} \end{array} \leftarrow c_B^T B^{-1} b.$$

Note: $\tilde{x}$ is feasible $\Leftrightarrow$ $B^{-1} b \geq 0$

$\tilde{x}$ is optimal $\Leftrightarrow$ $\sigma \geq c^T$

Note σ = obj. row + c^T = $(\sigma_1, \sigma_2, \ldots, \sigma_s)$

Notation in Book: $\sigma = (z_1, z_2, \ldots, z_s)$.

Sensitivity Analysis:

Assume $\tilde{x}$ is an optimal solution to Problem P.

Caveat: A, b, c may contain approximate values!

What happens if they change?

How much can they change before $\tilde{x}$ is no longer feasible/optimal?

Today: Examine this when single entry of b or c changes.

Assume c_ℓ is replaced with $c_\ell + \Delta c_\ell$

c is replaced by $\hat{c} = c + \Delta c_\ell \, e_\ell \in \mathbb{R}^s$, $e_\ell = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ ℓ-th entry

Problem $\hat{P}$:

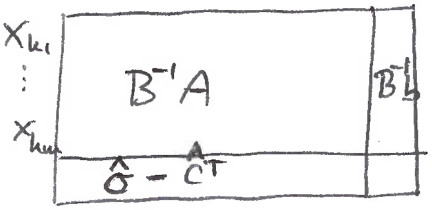
Maximize $\hat{z} = \hat{c}^T x$

subject to $Ax = b$; $x \geq 0$ in $\mathbb{R}^s$.

Constraints are the same as for P, so $\tilde{x}$ is a basic feasible solution for $\hat{P}$.

Q: Is $\tilde{x}$ optimal for $\hat{P}$?

$\hat{\sigma} = \hat{c}_B^T B^{-1} A$.



$\tilde{x}$ optimal for $\hat{P}$ $\Leftrightarrow$ $\hat{\sigma} \geq \hat{c}^T$.

Remark: If $\tilde{x}$ not optimal, then use simplex algorithm to find new optimal solution.

Case 1: Assume x_ℓ is NOT a basic variable.

Then $\hat{c}_B = c_B$

$$\hat{\sigma} = c_B^T B^{-1} A = \sigma$$

$$\tilde{x} \text{ optimal for } \hat{P} \iff \sigma \geq \hat{c}^T$$

$$\iff \sigma_\ell \geq c_\ell + \Delta c_\ell$$

$$\iff \Delta c_\ell \leq \sigma_\ell - c_\ell$$

$\therefore$ Can increase c_ℓ by as much as ℓ-th entry of objective row in final tableau, and $\tilde{x}$ still optimal.

Case 2: $\ell = k_i$, $x_\ell = x_{k_i}$ basic variable of row i.

$$\hat{c} = c + \Delta c_\ell e_\ell \in \mathbb{R}^s$$

$$\hat{c}_B = c_B + \Delta c_\ell e_i \in \mathbb{R}^m, \qquad e_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i\text{-th entry.}$$

$$\hat{\sigma} = \hat{c}_B^T B^{-1} A = c_B^T B^{-1} A + \Delta c_\ell \, e_i^T B^{-1} A$$

$$= \sigma + \Delta c_\ell \cdot R_i$$

where $R_i = e_i^T B^{-1} A = i$-th row of $B^{-1}A$.

$$B^{-1}A = \begin{bmatrix} R_1 \\ \hline R_2 \\ \hline \vdots \\ \hline R_m \end{bmatrix}$$

$$\boxed{\begin{array}{l} \tilde{x} \text{ optimal for } \hat{P} \\ \Updownarrow \\ \hat{\sigma} \geq \hat{c}^T \end{array}}$$

Let $j = k_r$ be index of basic variable x_{k_r}.

Then $\sigma_j = c_j$, since obj. row contains $\sigma_j - c_j = 0$.

If $j \neq \ell$ then:

$$\hat{\sigma}_j = \sigma_j + \Delta c_\ell \cdot R_{ij} = c_j + 0 = \hat{c}_j$$

If $j = \ell$ then:

$$\hat{\sigma}_\ell = \sigma_\ell + \Delta c_\ell \cdot R_{i\ell} = c_\ell + \Delta c_\ell = \hat{c}_\ell$$

$\therefore$ If j is index of basic variable, then $\hat{\sigma}_j = \hat{c}_j$.

Assume $1 \leq j \leq s$, x_j is NOT a basic variable.

$$\hat{\sigma}_j \geq \hat{c}_j \iff \sigma_j + \Delta c_\ell \cdot R_{ij} \geq c_j$$

$$\iff \sigma_j - c_j \geq -\Delta c_\ell \cdot R_{ij} \qquad ⊛$$

If $R_{ij} = 0$: ⊛ true for any Δc_ℓ.

If $R_{ij} > 0$: ⊛ true for $\Delta c_\ell \geq -\frac{\sigma_j - c_j}{R_{ij}}$

If $R_{ij} < 0$: ⊛ true for $\Delta c_\ell \leq -\frac{\sigma_j - c_j}{R_{ij}}$

Note: $\tilde{x}$ optimal for $\hat{P}$

$\Updownarrow$

⊛ holds for all j with x_j NON-basic.

Thm Assume $\hat{c} = c + \Delta c_\ell \cdot e_\ell$ where $x_\ell = x_{k_i}$ basic variable of row i.

Then $\tilde{x}$ is optimal for $\hat{P}$

$\Updownarrow$

$$\max_j \left\{ -\frac{\sigma_j - c_j}{R_{ij}} \;\middle|\; R_{ij} > 0 \right\} < \Delta c_\ell < \min_j \left\{ -\frac{\sigma_j - c_j}{R_{ij}} \;\middle|\; R_{ij} < 0 \right\}$$

where j runs over all indices of non-basic variables.

Note: Compute all $-\frac{(\text{obj row})_j}{R_{ij}}$ for which $R_{ij} \neq 0$. $\max\{-\} \leq \Delta c_\ell \leq \min\{+\}$

Example Consider linear problem P encoded by:

$$\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
x_6 & 0 & -1 & -2 & -1 & 0 & 1 & -3 \\
x_1 & 1 & -2 & -5 & -3 & 0 & 0 & -7 \\
x_5 & 0 & 7 & 17 & 8 & 1 & 0 & 26 \\
\hline
 & 0 & 8 & 19 & 12 & 0 & 0 & 33
\end{array}$$

Maximize $z = c^T x$

subj. to $Ax = b$, $x \geq 0$

$z = -8x_2 - 19x_3 - 12x_4 + 33$

$c^T = (0, -8, -19, -12, 0, 0)$

Find tableau representing optimal solution $\tilde{x}$ by using e.g. dual simplex algorithm:

$$\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
x_2 & 2 & 1 & 0 & -1 & 0 & -5 & 1 \\
x_3 & -1 & 0 & 1 & 1 & 0 & 2 & 1 \\
x_5 & 3 & 0 & 0 & -2 & 1 & 1 & 2 \\
\hline
 & 3 & 0 & 0 & 1 & 0 & 2 & 6
\end{array}$$

Let $\hat{P}$ be problem obtained by replacing c_ℓ with $c_\ell + \Delta c_\ell$.

Condition for $\tilde{x}$ optimal solution to $\hat{P}$:

$$\left.\begin{array}{ll}
\ell = 1: & -\infty < \Delta c_1 \leq 3 \\
\ell = 4: & -\infty < \Delta c_4 \leq 1 \\
\ell = 6: & -\infty < \Delta c_6 \leq 2
\end{array}\right\} \quad x_\ell \text{ non-basic.}$$

$\ell = 2$: x_2 basic variable of row 1.

Compute: $-\frac{3}{2}$, $-\frac{1}{-1}$, $-\frac{2}{-5}$

$$\max\left\{-\frac{3}{2}\right\} \leq \Delta c_2 \leq \min\left\{\frac{1}{1}, \frac{2}{5}\right\}$$

$$-\frac{3}{2} \leq \Delta c_2 \leq \frac{2}{5}$$

$\ell = 3$: X_3 basic variable of row 2.

Compute: $-\frac{3}{-1}$, $-\frac{1}{1}$, $-\frac{2}{2}$

$$\max\left\{-\frac{1}{1}, -\frac{2}{2}\right\} \le \Delta c_3 \le \min\left\{\frac{3}{1}\right\}$$

$$-1 \le \Delta c_3 \le 3$$

$\ell = 5$: X_5 basic variable of row 3.

Compute: $-\frac{3}{3}$, $-\frac{1}{-2}$, $-\frac{2}{1}$

$$\max\left\{-\frac{3}{3}, -\frac{2}{1}\right\} \le \Delta c_5 \le \min\left\{\frac{1}{2}\right\}$$

$$-1 \le \Delta c_5 \le \frac{1}{2}$$

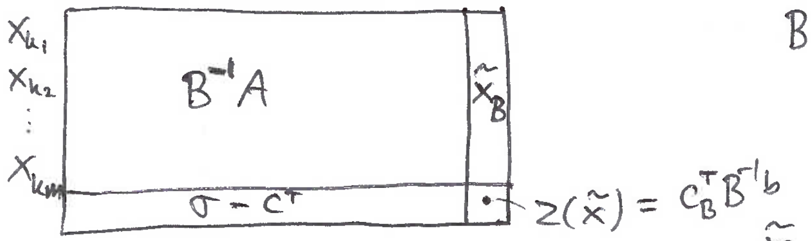
Note: 1) No cases with $R_{ij} = 0$ encountered. in this example.

2) If no positive values $-\frac{(\text{obj row})_j}{R_{ij}}$, then $\Delta c_\ell \le +\infty$

3) If no negative values $-\frac{(\text{obj row})_j}{R_{ij}}$, then $-\infty < \Delta c_\ell$.

<u>Problem P</u>: Maximize $z = c^T x$
subject to $Ax = b$, $x \geq 0$ in $\mathbb{R}^s$.
A $m \times s$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^s$.

Let $x_{k_1}, x_{k_2}, \dots, x_{k_m}$ be basic variables of basic solution $\tilde{x} \in \mathbb{R}^s$.
Tableau representing $\tilde{x}$:



$$B = \begin{bmatrix} A_{k_1} & A_{k_2} & \cdots & A_{k_m} \end{bmatrix} \qquad c_B = \begin{bmatrix} c_{k_1} \\ c_{k_2} \\ \vdots \\ c_{k_m} \end{bmatrix}$$

$$\tilde{x}_B = B^{-1} b, \quad \sigma = c_B^T B^{-1} A$$

<u>Note:</u> $\tilde{x}$ <u>feasible</u> $\Leftrightarrow$ $\tilde{x}_B = B^{-1} b \geq 0$

$\tilde{x}$ <u>optimal</u> $\Longleftarrow$ $\sigma = c_B^T B^{-1} A \geq c^T$

Assume $\tilde{x}$ optimal feasible solution, so that $B^{-1}b \geq 0$, $\sigma \geq c^T$.

<u>Changes to resource vector b:</u>

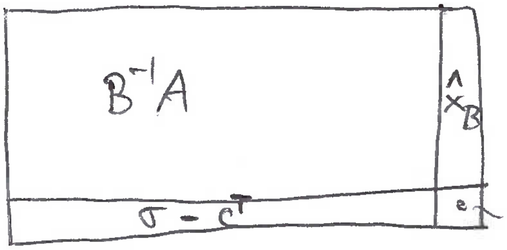
Assume b_ℓ is replaced with $b_\ell + \Delta b_\ell$.
b is replaced with $\hat{b} = b + \Delta b_\ell \cdot e_\ell \in \mathbb{R}^m$, $e_\ell = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \ell\text{-th entry}$

<u>Problem $\hat{P}$</u>: Maximize $z = c^T x$
subject to $Ax = \hat{b}$, $x \geq 0$ in $\mathbb{R}^s$.

Q: Is $\tilde{x}$ still an optimal feasible solution?
A: No! $\tilde{x}$ is NOT a feasible solution to $\hat{P}$
since $A\tilde{x} = b \neq \hat{b}$ (if $\Delta b_\ell \neq 0$.)

Q: Does $\hat{P}$ have an optimal basic feasible solution $\hat{x}$
that uses same basic variables $x_{k_1}, x_{k_2}, \dots, x_{k_m}$?

Tableau for $\hat{P}$ representing $\hat{x}$:



$\hat{x}_B = B^{-1}\hat{b}$

$z(\hat{x}) = c_B^T B^{-1}\hat{b}$

Note: $\sigma = c_B^T B^{-1} A$ does not depend on b, so $\hat{x}$ is optimal for $\hat{P}$.

Q: Is $\hat{x}$ feasible solution to $\hat{P}$?

$\hat{x}$ satisfies constraint $A\hat{x} = \hat{b}$. (by construction.)

Is $\hat{x} \geq 0$ in $\mathbb{R}^s$?

$$\hat{x}_B = B^{-1}\hat{b} = B^{-1}(b + \Delta b_\ell e_\ell) = B^{-1}b + \Delta b_\ell B^{-1} e_\ell$$

$$\hat{x}_B = \tilde{x}_B + \Delta b_\ell B^{-1} e_\ell$$

∴ $\hat{x}_B$ feasible solution to $\hat{P}$ ⇔ $\tilde{x}_B + \Delta b_\ell B^{-1} e_\ell \geq 0$

Note: $B^{-1}e_\ell$ = ℓ-th column of B^{-1}.

Note: Assume A has pivot in row ℓ, column j:

$A = [A_1\ A_2 \cdots A_j \cdots A_s]$, $A_j = e_\ell$

$B^{-1}A = [B^{-1}A_1\ B^{-1}A_2 \cdots B^{-1}A_j \cdots B^{-1}A_s]$

$B^{-1}e_\ell = B^{-1}A_j$ = column j of $B^{-1}A$ – can be found in final tableau for $\tilde{x}$.

Example Problem $\mathcal{P}$ encoded by

$$\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
x_6 & 0 & -1 & -2 & -1 & 0 & 1 & -3 \\
x_1 & 1 & -2 & -5 & -3 & 0 & 0 & -7 \\
x_5 & 0 & 7 & 17 & 8 & 1 & 0 & 26 \\
\hline
 & 0 & 8 & 19 & 12 & 0 & 0 & 33
\end{array}$$

Maximize $z = 33 - 8x_2 - 19x_3 - 12x_4$

subj. to $Ax = b, \quad x \geq 0$

$$b = \begin{bmatrix} -3 \\ -7 \\ 26 \end{bmatrix}$$

Optimal solution $\tilde{x}$ encoded by:

$$\begin{array}{c|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
x_2 & 2 & 1 & 0 & -1 & 0 & -5 & 1 \\
x_3 & -1 & 0 & 1 & 1 & 0 & 2 & 1 \\
x_5 & 3 & 0 & 0 & -2 & 1 & 1 & 2 \\
\hline
 & 3 & 0 & 0 & 1 & 0 & 2 & 6
\end{array}$$

$\tilde{x} = (0, 1, 1, 0, 2, 0)^T$

$$\tilde{x}_B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Let $\hat{\mathcal{P}}$ be problem obtained by replacing $b_\ell + \Delta b_\ell$.

Does $\hat{\mathcal{P}}$ have optimal BFS using same basic variables x_2, x_3, x_5?

$\ell = 1$: $B^{-1}e_1 = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} =$ column 6 of $B^{-1}A$.

$$\tilde{x}_B + \Delta b_1 \cdot B^{-1}e_1 \geq 0$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \Delta b_1 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \geq 0 \qquad \Delta b_1 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \geq -\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Compute: $-\frac{1}{-5}, \; -\frac{1}{2}, \; -\frac{2}{1}$

$$\max\left\{-\frac{1}{2}, -\frac{2}{1}\right\} \leq \Delta b_1 \leq \min\left\{\frac{1}{5}\right\}$$

$$-\frac{1}{2} \leq \Delta b_1 \leq \frac{1}{5}$$

$l = 2$: $B^{-1}e_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ = column 1 of $B^{-1}A$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \Delta b_2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \geq 0$$

Compute : $-\frac{1}{2}, \; -\frac{1}{-1}, \; -\frac{2}{3}$

$$\max\left\{-\frac{1}{2}, -\frac{2}{3}\right\} \leq \Delta b_2 \leq \min\left\{\frac{1}{1}\right\}$$

$$-\frac{1}{2} \leq \Delta b_2 \leq 1$$

$l = 3$: $B^{-1}e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ = column 5 of $B^{-1}A$.

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \Delta b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \geq 0$$

Compute : $\cancel{-\frac{1}{0}}, \; \cancel{-\frac{1}{0}}, \; -\frac{2}{1}$

$$\max\left\{-\frac{2}{1}\right\} \leq \Delta b_3 \leq \min\{\;\}$$

$$-2 \leq \Delta b_3 < \infty.$$

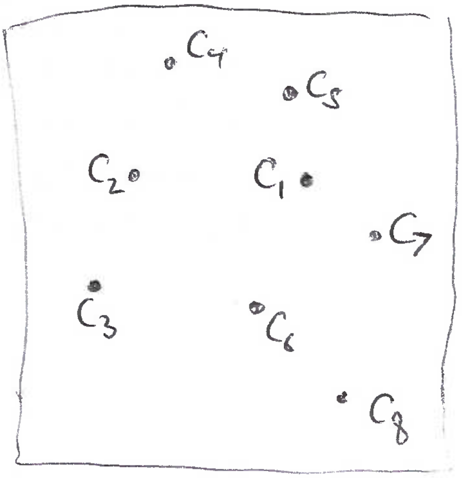
Traveling Salesman Problem

Must visit n cities $C_1, C_2, \dots, C_n$.

Start at C_1, visit other cities once each, return to C_1.

Order of other cities does not matter.

Want: Tour of smallest total distance.



Set c_{ij} = distance from C_i to C_j.

Tour determined by variables x_{ij}, $i,j \in [1,n]$:

$$x_{ij} = \begin{cases} 1 & \text{if } C_i \to C_j \text{ (direct connection)} \\ 0 & \text{else.} \end{cases}$$

Example $C_1 \to C_3 \to C_6 \to C_4 \to C_2 \to C_5 \to C_7 \to C_8 \to C_1$

$$x_{13} = x_{36} = x_{64} = \dots = x_{81} = 1.$$

Constraints:

Exactly one city after C_i: $\sum_{j=1}^{n} x_{ij} = 1$. $1 \le i \le n$

Exactly one city before C_j: $\sum_{i=1}^{n} x_{ij} = 1$. $1 \le j \le n$.

First attempt:

Minimize $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

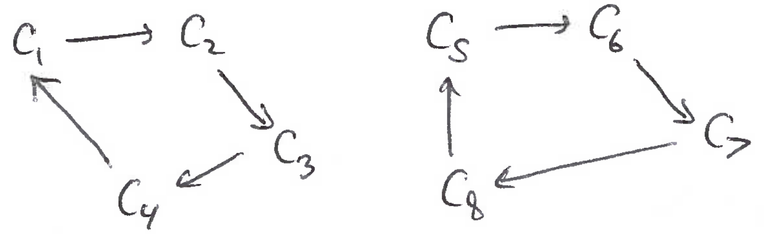
Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } 1 \le i \le n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } 1 \le j \le n.$$

$$x_{ij} \in \{0,1\} \quad \text{for } i,j \in [1,n].$$

Problem:



OR: $C_1 \circlearrowright \quad C_2 \circlearrowright \quad C_3 \circlearrowright \quad \cdots \quad C_n \circlearrowright$

$x_{11} = x_{22} = \cdots = x_{nn} = 1$. $x_{ij} = 0$ for $i \neq j$.

Fix: Introduce $n-1$ new variables: $u_2, u_3, \dots, u_n$.

$(n-1)^2 - (n-1)$ new constraints:

$$\circledast \qquad u_i - u_j + n x_{ij} \le n-1 \quad \text{for } i,j \in [2,n],\ i \neq j.$$

Claim: The old and new constraints can be satisfied exactly when $\{x_{ij}\}$ describes a single connected tour through all n cities.

Assume $\{x_{ij}\}$ describe a subtour $C_{k_1} \to C_{k_2} \to \cdots \to C_{k_r} \to C_{k_1}$ which does not include C_1.

length $= r$.

Add up constraints $\circledast$ for $(i,j) = (k_t, k_{t+1})$, $t = 1, 2, \dots, r$:

Since $x_{ij} = 1$ for each t, we get:

$$nr \le (n-1)r$$

Which is impossible!

∎

Now let $C_1 = C_{k_1} \to C_{k_2} \to \cdots \to C_{k_n} \to C_{k_{n+1}} = C_1$ be a single connected tour of all n cities.

Must show this is allowed by constraints.

Set $$x_{ij} = \begin{cases} 1 & \text{if } (i,j) = (k_t, k_{t+1}) \text{ for some } t \\ 0 & \text{else.} \end{cases}$$

Set $u_{k_t} = t$ for $t = 2, 3, 4, \dots, n$.

I.e. $u_i = t$ if C_i is the t-th city in tour.

Claim: ⊛ holds.

Let $i, j \in [2, n]$, $i \neq j$.

If $x_{ij} = 0$: Since $u_i, u_j \in [2, n]$ we have $|u_i - u_j| \leq n-2$.

$$\Rightarrow u_i - u_j + n x_{ij} \leq n-1.$$

If $x_{ij} = 1$: $(i,j) = (k_t, k_{t+1})$ for some t.

$u_i = t$, $u_j = t+1$.

$$u_i - u_j + n x_{ij} = t - (t+1) + n = n-1.$$

Problem (Traveling Salesman)

Minimize $z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } 1 \leq i \leq n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } 1 \leq j \leq n$$

$$u_i - u_j + n x_{ij} \leq n-1 \quad \text{for } i, j \in [2, n],\ i \neq j.$$

$$x_{ij} \in \{0, 1\} \quad \text{for } i, j \in [1, n]$$

optional constraint: $u_i \geq 0$ $\quad 2 \leq i \leq n$.

$u_i \geq 0$ and $u_i \in \mathbb{Z}$

Read about: Stock Cutting Problem
Fixed charge Problem
Knapsack Problem
Assignment Problem.

Either-Or Problem

Assume we need one of two constraints satisfied:

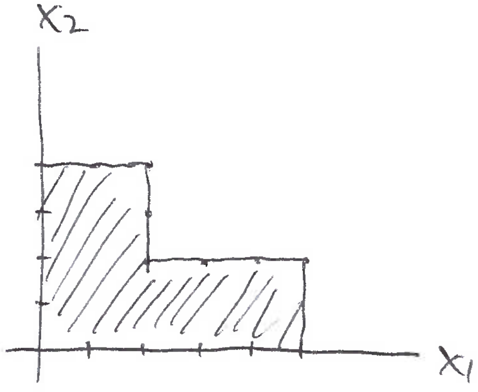
$$\sum_{j=1}^{n} a_{1j} x_j \le b_1 \qquad \text{OR} \qquad \sum_{j=1}^{n} a_{2j} x_j \le b_2$$

Example

Maximize $z = 3x_1 + 4x_2$

subject to

$$x_1 \le 5$$
$$x_2 \le 4$$
$$x_1 \le 2 \quad \text{OR} \quad x_2 \le 2$$
$$x_1, x_2 \ge 0$$



Note: ∘ NOT linear problem.
∘ Set of Feasible solutions NOT convex!

Reformulate as mixed integer problem:

Note $x_1 \le 10$ and $x_2 \le 10$ for all feasible solutions.

Introduce new variable $y \in \{0,1\}$.

New constraints:

$$x_1 - 2 \le 10y$$
$$x_2 - 2 \le 10(1-y)$$

Note: If $y = 0$ then $x_1 \le 2$, and $x_2 - 2 \le 10(1-y)$ redundant

if $y = 1$ then $x_2 \le 2$, $x_1 - 2 \le 10y$ redundant.

Problem:

$$\text{Maximize} \quad z = 3x_1 + 4x_2$$

$$\text{Subject to} \quad x_1 \le 5$$
$$x_2 \le 4$$
$$x_1 - 2 \le 10y$$
$$x_2 - 2 \le 10 - 10y$$
$$x_1, x_2 \ge 0$$
$$y \in \{0,1\}.$$

General situation:

$$\sum_{j=1}^{n} a_{1j} x_j \le b_1 \quad \text{or} \quad \sum_j a_{2j} x_j \le b_2 .$$

Assume we can find $M_1, M_2 \in \mathbb{R}$ such that

$$\sum a_{1j} x_j - b_1 \le M_1$$
$$\sum a_{2j} x_j - b_2 \le M_2$$

for all feasible solutions.

Use new variable $y \in \{0,1\}$.

Constraints:

$$\sum_j a_{1j} x_j - b_1 \le M_1 y$$
$$\sum_j a_{2j} x_j - b_2 \le M_2 (1-y).$$
$$y \in \{0,1\}.$$

Linear Opt. 11/24/2015 ①

Questions from one of you

1) Solve primal + dual, and MT2 Problem 7.

Primal Problem in canonical form :

Maximize $z = c^T x$

subject to $Ax = b$, $x \geq 0$ in $\mathbb{R}^s$

A $m \times s$

$b \in \mathbb{R}^m$, $c \in \mathbb{R}^s$.

Dual Problem :

Minimize $z' = b^T w$

subject to $A^T w \geq c$, $w \in \mathbb{R}^m$ unrestricted.

Method :

(1) Solve Primal problem with 2-phase algorithm. KEEP artificial variables in Phase 2.

(2) Compute $\sigma =$ final obj. row $+ (c^T, \underbrace{0, 0, \ldots, 0}_{\text{artificial vars.}})$

(3) Compute $\hat{w} = (\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_m})^T$ where $k_1, k_2, \ldots, k_m$ are indices of basic variables of initial tableau of Phase 1.

(4) Obtain w from $\hat{w}$ by changing the sign of $\hat{w}_i$ whenever we changed the sign of row i in $Ax = b$ to get started on Phase 1.

(i.e. $w_i = \begin{cases} \hat{w}_i & \text{if } b_i \geq 0 \\ -\hat{w}_i & \text{if } b_i < 0. \end{cases}$)

<u>Example</u> Problem P

Maximize $z = 3x_1 + x_2 - 2x_3 + 2x_4 - 2x_5 + x_6$

Subject to

$$\begin{array}{lcl} 2x_1 + x_5 - x_6 &=& 4 \\ x_1 + x_3 - x_4 &=& 1 \\ 2x_1 + x_2 + x_3 + x_6 &=& 15 \\ 2x_1 - x_3 - x_4 - x_6 &=& -5 \end{array}$$

$$\boxed{x \geq 0 \text{ in } \mathbb{R}^6}$$

<u>Dual Problem $P^{\vee}$</u>

Minimize $z' = 4w_1 + w_2 + 15w_3 - 5w_4$

subj. to

$$\begin{array}{lcl} 2w_1 + w_2 + 2w_3 + 2w_4 &\geq& 3 \\ w_3 &\geq& 1 \\ w_2 + w_3 - w_4 &\geq& -2 \\ -w_2 - w_4 &\geq& 2 \\ w_1 &\geq& -2 \\ -w_1 + w_3 - w_4 &\geq& 1 \end{array}$$

$$\boxed{w \in \mathbb{R}^4 \text{ unrestricted}}$$

To use 2-Phase algorithm, we have to replace P with:

<u>Problem $\tilde{P}$</u> Maximize $z = 3x_1 + x_2 - 2x_3 + 2x_4 - 2x_5 + x_6$

subj. to

$$\begin{array}{lcl} 2x_1 + x_5 - x_6 &=& 4 \\ x_1 + x_3 - x_4 &=& 1 \\ 2x_1 + x_2 + x_3 + x_6 &=& 15 \\ -2x_1 + x_3 + x_4 + x_6 &=& 5 \end{array}$$

$$\boxed{x \geq 0 \text{ in } \mathbb{R}^6}$$

<u>Note:</u> $\tilde{P}$ has same solution as P.

<u>But</u> Dual of $\tilde{P}$ is NOT $P^{\vee}$. $\tilde{P}^{\vee} \neq P^{\vee}$.

$\tilde{P}^{\vee}$ is obtained from $P^{\vee}$ by <u>changing sign of w_4</u> because we changed sign of constraint #4 in P!

Almost
Initial tableau of Phase 1:

	x_1	x_2	x_3	x_4	x_5	x_6	~~x_7~~	x_8	
x_5	2	0	0	0	1	-1	0	0	4
~~x_7~~	1	0	1	-1	0	0	1	0	1
x_2	2	1	1	0	0	1	0	0	15
~~x_8~~	-1	0	1	1	0	1	0	1	5
	0	0	0	0	0	0	1	1	0

Artificial variables : x_7, x_8
" "
y_1 y_2

Final tableau of Phase 2:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_1	1	2/7	0	0	1/7	0	-1/7	-1/7	4
x_3	0	-1/7	1	0	3/7	0	4/7	4/7	3
x_6	0	4/7	0	0	-5/7	1	-2/7	-2/7	4
x_4	0	1/7	0	1	4/7	0	-4/7	3/7	6
	0	1	0	0	2	0	-3	-1	22

optimal solution to P and $\tilde{P}$: $\tilde{x} = (4, 0, 3, 6, 0, 4)$.

$c^T = (3, 1, -2, 2, -2, 1, \underbrace{0, 0}_{\text{extra}})$

σ = obj. row + c^T = $(3, 2, -2, 2, 0, 1, -3, -1)$

Initial basic vars : x_5, x_7, x_2, x_8

$\hat{w} = (0, -3, 2, -1)$

$\tilde{w} = (0, -3, 2, 1)$. Sol. to P^v.

Check: $\tilde{x}$ feasible to P
$\tilde{w}$ feasible to P^v

$Z(\tilde{x}) = 22 = Z'(\tilde{w})$.

Why does this work?

Assume that $b \geq 0$, so that we don't change signs of any rows.

Initial tableau of Phase 1 (almost!):

$$\begin{array}{c|c|c|c|}
 & x_1 \; x_2 \quad \cdots \quad x_s & x_{s+1} \cdots x_{s+t} & \\
\hline
x_{v_1} & & & \\
x_{v_2} & A & J & b \\
\vdots & & & \\
x_{v_m} & & & \\
\hline
 & 0 \; 0 \; 0 \; \cdots \; 0 & 1 \; 1 \cdots 1 & \\
\hline
\end{array}$$

Initial basic variables $x_{v_1} \cdots x_{v_m}$.

$$\hat{A} = \left|\begin{array}{c|c} A & J \end{array}\right| \qquad \hat{A}_{v_i} = e_i$$

Tableau rep. problem P (with extra artificial vars):

$$\begin{array}{|c|c|c|}
x_1 \; x_2 \quad \cdots \quad x_s & x_{s+1} \cdots x_{s+t} & \\
\hline
A & J & b \\
\hline
-c^T & 0 \, 0 \cdots 0 & 0 \\
\hline
\end{array}$$

Assume optimal solution $\tilde{x}$ has basic variables $x_{k_1}, \dots, x_{k_m}$.

$$B = \left[\begin{array}{c|c|c} \hat{A}_{k_1} & \cdots & \hat{A}_{k_m} \end{array}\right] \qquad c_B = \begin{bmatrix} c_{k_1} \\ \vdots \\ c_{k_m} \end{bmatrix}$$

$$\boxed{\begin{array}{l} \tilde{w}^T = c_B^T B^{-1} \in \mathbb{R}^m \\ \text{optimal solution to} \\ \text{dual problem.} \end{array}}$$

Final tableau of Phase 2:

$$\begin{array}{|c|c|c|}
\hline
B^{-1}A & B^{-1}J & B^{-1}b \\
\hline
c_B^T B^{-1} A - c^T & c_B^T B^{-1} J & \\
\hline
\end{array}$$

$$\tilde{x}_B = B^{-1} b$$

$$\boxed{c^T \tilde{x} = c_B^T \tilde{x}_B = \tilde{w}^T b = b^T \tilde{w}}$$

$$\boxed{c^T \tilde{x} = c_B^T \tilde{x}_B = \tilde{w}^T b = b^T \tilde{w}}$$

Note:

~~obj. row: $\sigma - (c^T, \underbrace{0, 0, \dots, 0}_{t})$ where~~

$$\sigma = \text{obj. row} + (c^T, 0, 0, \dots, 0) = c_B^T B^{-1} \hat{A}.$$

$\tilde{w}^T = c_B^T B^{-1} \in \mathbb{R}^m$ optimal sol. to dual problem.

$$\tilde{w}_i = c_B^T B^{-1} e_i = c_B^T B^{-1} \hat{A}_{v_i} = \sigma_{v_i}$$

Sensitivity Analysis vs Phase 2

Maximize $z = -8x_2 - 19x_3 - 12x_4$
subject to
$$\begin{aligned} x_1 + 4x_2 + 10x_3 + 4x_4 + x_5 &\leq 16 \\ -x_1 + 2x_2 + 5x_3 + 3x_4 &= 7 \\ 7x_2 + 17x_3 + 8x_4 + x_5 &= 26 \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	u_1	y_1	y_2	
x_6	1	4	10	4	1	1	0	0	16
y_1	-1	2	5	3	0	0	1	0	7
y_2	0	7	17	8	1	0	0	1	26
	0	0	0	0	0	0	1	1	0

← (Almost) init. tab Phase 1.

Final Phase 2:

	x_1	x_2	x_3	x_4	x_5	u_1	y_1	y_2	
x_3	-1	0	1	1	0	2	3	-2	1
x_2	2	1	0	-1	0	-5	-7	5	1
x_5	3	0	0	-2	1	1	-2	0	2
	3	0	0	1	0	2	-1	-2	-27

$$\tilde{x} = [0, 1, 1, 0, 2]^T$$

$$z = c^T x, \qquad c = \begin{bmatrix} 0 \\ -8 \\ -19 \\ -12 \\ 0 \end{bmatrix}$$

Q: Is $\tilde{x}$ still optimal solution if c_ℓ replaced with $c_\ell + \Delta c_\ell$?

$\ell = 1$: X_1 NOT basic variable in final tableau.

$$\Delta c_1 \leq (\text{final obj. row})_1 = 3$$

$\ell = 2$: X_2 basic var. of row 2.

Compute: $-\frac{3}{2}$, 1 $-\frac{1}{-1}$, $-\frac{2}{-5}$,

$$-\frac{3}{2} \leq \Delta c_2 \leq \frac{2}{5}$$

Include slack variables but NOT artificial vars!

$$b = \begin{bmatrix} 16 \\ 7 \\ 26 \end{bmatrix} \qquad X_B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Q: Does problem still have optimal sol with basic vars x_2, x_3, x_5 if b_ℓ is replaced with $b_\ell + \Delta b_\ell$?

A: yes $\Leftrightarrow$ $\tilde{X}_B + \Delta b_\ell \cdot B^{-1} e_\ell \geq 0$.

~~$B^{-1}e_2$~~

$\ell = 1$: $B^{-1} e_1 =$ column of a_1 $= \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

$$\tilde{X}_B + \Delta b_1 \cdot B^{-1} e_1 \geq 0$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \Delta b_1 \cdot \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \geq 0$$

Compute: $-\frac{1}{2}$, $-\frac{1}{-5}$, $-\frac{2}{1}$

$$-\frac{1}{2} \leq \Delta b_1 \leq \frac{1}{5}$$

$\ell = 2$: $B^{-1} e_2 =$ column of Y_1 $= \begin{bmatrix} 3 \\ -7 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \Delta b_2 \begin{bmatrix} 3 \\ -7 \\ -2 \end{bmatrix} \geq 0$$

Compute: $-\frac{1}{3}$, $-\frac{1}{-7}$, $-\frac{2}{-2}$

$$-\frac{1}{3} \leq \Delta b_2 \leq \frac{1}{7}.$$

$\ell = 3$: $B^{-1} e_3$ = column of r_2 = $\begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \Delta b_3 \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} \geq 0$$

Compute : $-\frac{1}{-2}, \; -\frac{1}{5}, \; \cancel{-\frac{2}{0}}$

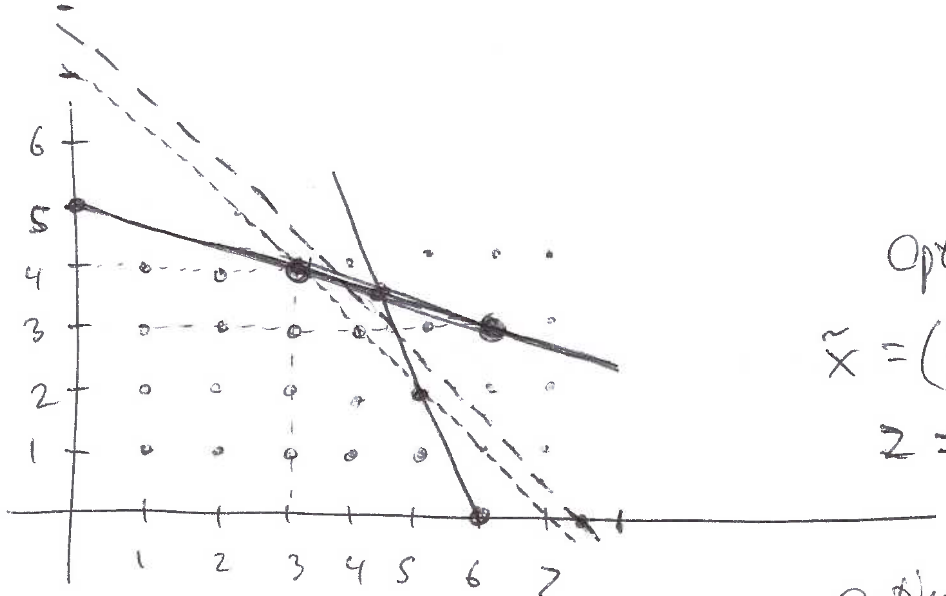
$$-\frac{1}{5} \leq \Delta b_3 \leq \frac{1}{2} .$$

Maximize $z = 3x_1 + 2x_2$

subject to $2x_1 + x_2 \le 12$

$x_1 + 3x_2 \le 15$

$(x_1, x_2) \ge 0$ in $\mathbb{Z}^2$.



Optimal sol. in $\mathbb{R}^2$:

$\tilde{x} = \left(\frac{21}{5}, \frac{18}{5}\right) = (4.2, 3.6)$

$z = \frac{99}{5} = 19.8$

Optimal sol. in $\mathbb{Z}^2$:

$\tilde{x}' = (5, 2)$.

$z = 19$.

First cutting plane:

$x_2 - u_1 \le 3$

$u_1 = 12 - 2x_1 - x_2$

Cutting plane:

$2x_1 + 2x_2 \le 15$

Second cutting plane:

$x_1 + u_1 - u_3 \le 4$

$u_3 = 3 + u_1 - x_2$

Cutting plane:

$x_1 + x_2 < 7$

Linear Opt. 11/25/2015

Integer & fractional parts

Def For $x \in \mathbb{R}$ define

$$[x] = \max\{ m \in \mathbb{Z} \mid m \le x \} \quad \text{integer part}$$

Fractional part is $x - [x]$.

Note: $0 \le x - [x] < 1$

Examples

$x = 4.75$ $\quad [4.75] = 4$ $\quad$ frac. part $= 0.74$

$x = 2$ $\quad [2] = 2$ $\quad$ frac part $= 0$

$x = -3.2$ $\quad [-3.2] = -4$ $\quad$ frac. part $= 0.8$

Pure Integer Programming

Consider single equation

$$x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = b$$

where $a_2, \to a_n, b \in \mathbb{R}$, $x_1, \to x_n \in \mathbb{Z}$, $x_i \ge 0$.

If eqn. satisfied by $x = (x_1, \to x_n)$, then we must have:

$$x_1 + [a_2] x_2 + [a_3] x_3 + \cdots + [a_n] x_n \le b$$

Now since LHS is an <u>integer</u>, we must have

$$x_1 + [a_2] x_2 + [a_3] x_3 + \cdots + [a_n] x_n \le [b]$$

Add new slack variable $u \in \mathbb{Z}$, $u \ge 0$:

$$x_1 + [a_2] x_2 + [a_3] x_3 + \cdots + [a_n] x_n + u = [b].$$

Note: Before $x = (b, 0, \to 0) \in \mathbb{R}^n$ was a non-integer sol.
This is NOT a solution to new equation!

Example from Book :

Maximize $z = 5x_1 + 6x_2$

Subject to

$$10x_1 + 3x_2 \leq 52$$

$$2x_1 + 3x_2 \leq 18$$

$(x_1, x_2) \geq 0$ in $\mathbb{Z}^2$.

Linear Opt. 12/1/2015 PURE INT PROG

Write
$\text{frac}(x) = x - [x]$

Recall: $[x] = \max\{m \in \mathbb{Z} \mid m \le x\}$

$[3.4] = 3$, $[-3.4] = -4$

Assume $x = (x_1, x_2, \ldots, x_n)$ satisfies

$x \in \mathbb{Z}^n$

$x \geq 0$

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$, $a_j, b \in \mathbb{R}$.

Then

$[a_1] x_1 + [a_2] x_2 + \cdots + [a_n] x_n \le [b]$ ⊛

write $g_j = a_j - [a_j]$ = frac part of a_j

$f = b - [b]$ = frac part of b.

⊛ $\Leftrightarrow$ $g_1 x_1 + g_2 x_2 + \cdots + g_n x_n \geq f$.

Example Assume $x \geq 0$ in $\mathbb{Z}^2$ and $x_1 + \frac{5}{3} x_2 = \frac{34}{3}$.

Then $\frac{2}{3} x_2 = \frac{1}{3}$ + integer

$\frac{2}{3} x_2 \geq \frac{1}{3}$.

This says $x_2 \geq$ [illegible] $\frac{1}{2}$. (So $x_2 \geq 1$.)

Note: $-x_1 - \frac{5}{3} x_2 = -\frac{34}{3}$

$\text{frac}\left(-\frac{5}{3}\right) = -\frac{5}{3} - (-2) = \frac{1}{3}$

$\text{frac}\left(-\frac{34}{3}\right) = -\frac{34}{3} - (-12) = \frac{2}{3}$.

Obtain: $\frac{1}{3} x_2 \geq \frac{2}{3}$.

This says $x_2 \geq 2$. Better!!!

Assume $x \geq 0$ in $\mathbb{Z}^n$ and

$$\sum_{j=1}^{n} a_j x_j = b$$

Then $$\sum_{j=1}^{n} (-a_j) x_j = -b$$

Assume $a_j, b \notin \mathbb{Z}$:

$\mathrm{frac}(-a_j) = 1 - g_j$

$\mathrm{frac}(-b) = 1 - f.$

This implies $$\sum_{j=1}^{n} (1-g_j) x_j \geq 1-f$$

$$\Leftrightarrow \sum_{j=1}^{n} \frac{f}{1-f}(1-g_j) x_j \geq f. \qquad (**)$$

Compare to $$\sum_{j=1}^{n} g_j x_j \geq f.$$

~~This inequality is strongest if coeff. of~~

Strongest inequality? Coefficient of x_j should be SMALL!

If $g_j \leq f$ then $g_j \leq \frac{f}{1-f}(1-g_j)$

If $g_j \geq f$ then $g_j \geq \frac{f}{1-f}(1-g_j)$.

Best of both worlds:

Thm Assume $x \geq 0$ in $\mathbb{Z}^n$ and satisfies $\sum_{j=1}^{n} a_j x_j = b$.
Set $g_j = \mathrm{frac}(a_j)$, $f = \mathrm{frac}(b)$.

$$d_j = \begin{cases} g_j & \text{if } g_j \leq f \\ \frac{f}{1-f}(1-g_j) & \text{if } g_j \geq f \end{cases}$$

Then we have $$\sum_{j=1}^{n} d_j x_j \geq f.$$

Example Maximize $z = 3x_1 + 2x_2$

subject to
$$2x_1 + x_2 \le 12$$
$$x_1 + 3x_2 \le 15$$
$$(x_1, x_2) \ge 0 \text{ in } \mathbb{Z}^2$$

Canonical form:
$$2x_1 + x_2 + u_1 = 12$$
$$x_1 + 3x_2 + u_2 = 15 \qquad x_j, u_j \in \mathbb{Z}, \ \ge 0.$$

From final tableau:
$$x_2 - \frac{1}{5}u_1 + \frac{2}{5}u_2 = \frac{18}{5}$$

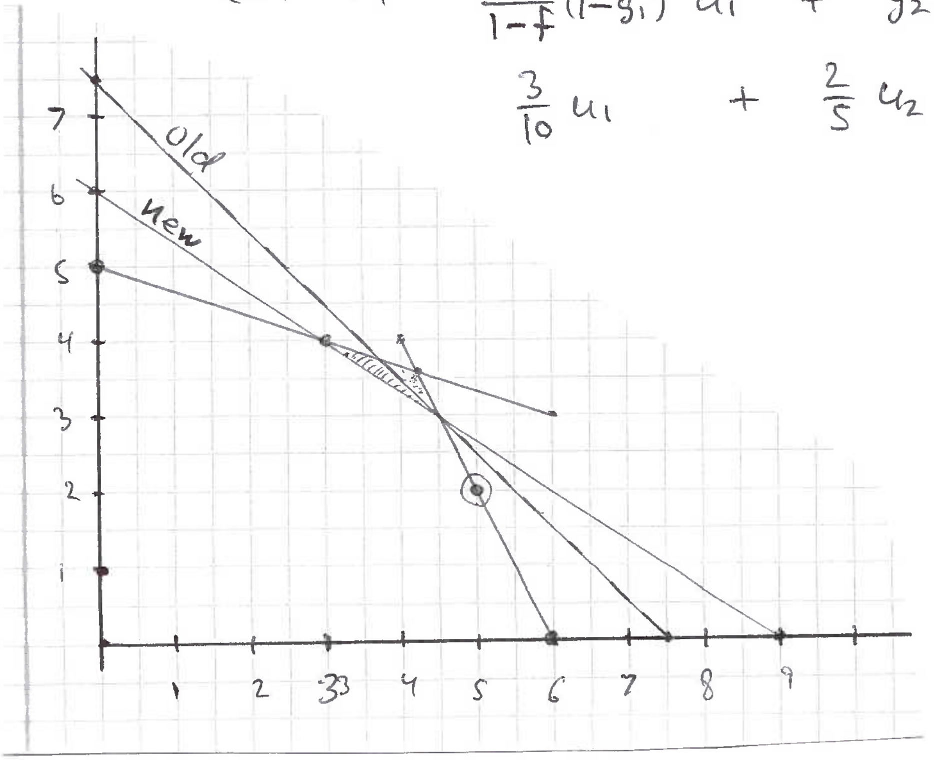
problem since $x_1 \in \mathbb{Z}$, $u_1 = u_2 = 0$ (not basic)

$$g_1 = \text{frac}\left(-\frac{1}{5}\right) = \frac{4}{5}$$
$$g_2 = \text{frac}\left(\frac{2}{5}\right) = \frac{2}{5}$$
$$f = \text{frac}\left(\frac{18}{5}\right) = 3/5 .$$

Last time: cut by ~~$x_2 - u_1 \le 3$~~ $g_1 u_1 + g_2 u_2 \ge f$

$\Updownarrow$
$$\frac{4}{5}u_1 + \frac{2}{5}u_2 \ge \frac{3}{5} \iff 2x_1 + 2x_2 \le 15$$

Then: Cut by
$$\frac{f}{1-f}(1-g_1)\, u_1 + g_2 u_2 \ge f$$
$$\frac{3}{10}u_1 + \frac{2}{5}u_2 \ge \frac{3}{5} \iff 2x_1 + 3x_2 \le 18$$



MIXED INT PROG

$x_j \in \mathbb{Z}$ for $j \in I$, $I \subseteq \{1, 2, \ldots, n\}$. subset.
$x_j \in \mathbb{R}$ for $j \notin I$.

Assume $x = (x_1, x_2, \ldots, x_n)$ satisfies

$$x \geq 0$$

$$x_1 \in \mathbb{Z}, \quad x_j \in \mathbb{R} \text{ for } j \geq 2$$

$$x_1 + \sum_{j=2}^{n} a_j x_j = b, \quad a_j, b \in \mathbb{R}, \quad b \notin \mathbb{Z}.$$

Then $\sum_{j=2}^{n} a_j x_j = f + \text{integer}$, $f = \text{frac}(b)$.

Assume also $a_j \geq 0 \ \forall j$:

$$\sum_{j=2}^{n} a_j x_j \geq f$$

Assume also $a_j \leq 0 \ \forall j$:

$$\sum_{j=2}^{n} a_j x_j \leq f - 1$$

$$\Updownarrow \quad \sum_{j=2}^{n} \left(-\frac{f}{1-f} a_j \right) x_j \geq f$$

Best of ALL worlds !!

<u>Thm</u> Assume $x = (x_1, x_2, \ldots, x_n)$ satisfies

$$x \geq 0$$

$$x_j \in \mathbb{Z} \text{ for } j \in I, \text{ where } I \subseteq \{1, 2, \ldots, n\}, \quad 1 \in I.$$

$$x_1 + \sum_{j=2}^{n} a_j x_j = b \quad \text{where } a_j, b \in \mathbb{R}, \quad b \notin \mathbb{Z}.$$

Set $g_j = \text{frac}(a_j)$, $f = \text{frac}(b)$.

$$d_j = \begin{cases} g_j & \text{if } j \in I, \ g_j \leq f \\ \frac{f}{1-f}(1-g_j) & \text{if } j \in I, \ g_j \geq f \\ a_j & \text{if } j \notin I, \ a_j \geq 0 \\ -\frac{f}{1-f} a_j & \text{if } j \notin I, \ a_j \leq 0. \end{cases}$$

Then $\sum_{j=2}^{n} d_j x_j \geq f.$

Cutting Plane Theorem

Assume $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ satisfies

- $x_j \geq 0$ for $1 \leq j \leq n$
- $x_j \in \mathbb{Z}$ for $j \in I$, where $I \subseteq \{1, 2, \dots, n\}$ subset
- $\sum_{j=1}^{n} a_j x_j = b$ where $a_j, b \in \mathbb{R}$.

Set $g_j = \text{frac}(a_j)$, $f = \text{frac}(b)$

$$d_j = \begin{cases} g_j & \text{if } j \in I \text{ and } g_j \leq f \\ \frac{f}{1-f}(1-g_j) & \text{if } j \in I \text{ and } g_j > f \\ a_j & \text{if } j \notin I \text{ and } a_j \geq 0 \\ \frac{f}{1-f}(-a_j) & \text{if } j \notin I \text{ and } a_j < 0. \end{cases}$$

Then $\sum_{j=1}^{n} d_j x_j \geq f$.

Proof $I^+ = \{ j \in I \mid g_j \le f \}$ ⑤.⑤

$I^- = \{ j \in I \mid g_j > f \}$

$N^+ = \{ j \in [1,n] \setminus I \mid a_j \ge 0 \}$

$N^- = \{ j \in [1,n] \setminus I \mid a_j < 0 \}$

Set $h = f - \sum_{j \in I^+} g_j x_j - \sum_{j \in N^+} a_j x_j$

If $h \le 0$ then the result is true.

WLOG $0 < h \le f < 1$.

$$\sum_{j \in I^-} a_j x_j + \sum_{j \in N^-} a_j x_j = b - \sum_{j \in I^+} a_j x_j - \sum_{j \in N^+} a_j x_j$$

$$\sum_{j \in I^-} g_j x_j + \sum_{j \in N^-} a_j x_j = h + \text{integer}$$

$$\sum_{j \in I^-} (1-g_j) x_j + \sum_{j \in N^-} (-a_j) x_j = (1-h) + \text{integer}$$

$$\sum_{j \in I^-} (1-g_j) x_j + \sum_{j \in N^-} (-a_j) x_j \geqslant (1-h)$$

$$\sum_{j \in I^-} \frac{h}{1-h} (1-g_j) x_j + \sum_{j \in N^-} \frac{h}{1-h} (-a_j) x_j \ge h$$

Since $\frac{h}{1-h} \le \frac{f}{1-f}$ we obtain

$$\sum_{j \in I^-} \frac{f}{1-f} (1-g_j) x_j + \sum_{j \in N^-} \frac{f}{1-f} (-a_j) x_j \ge f - \sum_{j \in I^+} g_j x_j - \sum_{j \in N^+} a_j x_j$$

as claimed.

☐

Example Maximize $z = x_1 + 2x_2 + x_3 + x_4$

subject to

$$2x_1 + x_2 + 3x_3 + x_4 \le 8$$

$$2x_1 + 3x_2 + 4x_4 \le 12$$

$$3x_1 + x_2 + 2x_3 \le 18$$

$$x \ge 0 \text{ in } \mathbb{R}^4.$$

$$x_1, x_3 \in \mathbb{Z}.$$

$$\frac{4}{9}x_1 + x_3 - \frac{1}{9}x_4 + \frac{1}{3}x_5 - \frac{1}{9}x_6 = \frac{4}{3}$$

$g_1 = \frac{4}{9}$ $g_3 = 0$ $g_4 = \frac{8}{9}$ $g_5 = \frac{1}{3}$ $g_6 = \frac{8}{9}$ $f = \frac{1}{3}$

$$\boxed{\frac{f}{1-f} = \frac{1}{2}}$$

$1 \in I$ and $g_1 > f$.

$$d_1 = \frac{f}{1-f}(1-g_1) = \frac{1/3}{1-1/3}\left(1 - 4/9\right) = \frac{1}{2}\frac{5}{9} = \frac{5}{18}$$

$d_3 = 0$ since $3 \in I$ and $g_3 = 0$

$4 \notin I$, $a_4 < 0$: $d_4 = -\frac{f}{1-f}a_4 = +\frac{1}{18} = +\frac{1}{18}$

$5 \notin I$, $a_5 > 0$: $d_5 = a_5 = \frac{1}{3}$.

$6 \notin I$, $a_6 < 0$: $d_6 = -\frac{f}{1-f}a_6 = \frac{1}{18}$

Cutting plane:

$$\frac{5}{18}x_1 + \frac{1}{18}x_4 + \frac{1}{3}x_5 + \frac{1}{18}x_6 \ge \frac{1}{3}.$$

Linear Opt. 12/4/2015

THE BRANCH AND BOUND METHOD FOR (MIXED) INTEGER PROGRAMS

Problem P:
Maximize $z = c^T x$ subject to
$Ax = b$
$x \geq 0$ in $\mathbb{R}^s$.
$x \in \mathbb{Z}$ for $j \in I$.
Here A is an $m \times s$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^s$, and $I \subset \{1, 2, \ldots, s\}$.
Feasible solutions: $S = \{x \in \mathbb{R}^s \mid Ax = b \text{ and } x \geq 0 \text{ and } x_j \in \mathbb{Z} \text{ for } j \in I\}$

Corresponding linear problem:
Problem $\widehat{P}$:
Maximize $z = c^T x$ subject to $Ax = b$ and $x \geq 0$ in $\mathbb{R}^s$.
Feasible solutions: $\widehat{S} = \{x \in \mathbb{R}^s \mid Ax = b \text{ and } x \geq 0\}$
Notice that $S \subset \widehat{S}$.

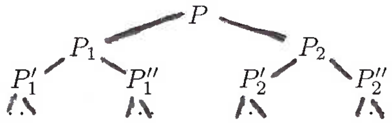
Let $\widehat{x} \in \widehat{S}$ be an optimal solution to the linear problem $\widehat{P}$.
This solution $\widehat{x}$ can be found efficiently using the simplex algorithm.

Observation 1:
If $\widehat{x} \in S$, then $\widehat{x}$ is an optimal solution the the mixed integer problem P.

Assume that $\widehat{x} \notin S$.
Observation 2:
If $\widetilde{x} \in S$ is any feasible solution to P, then $\widetilde{x} \in \widehat{S}$, so we have $z(\widetilde{x}) \leq z(\widehat{x})$.
Therefore $z(\widehat{x})$ is an **upper bound** for the optimal objective function value of P.

Idea of Branch and Bound:
Assume that $\widehat{x}_k \notin \mathbb{Z}$ for some $k \in I$.
The optimal solution $\widetilde{x} \in S$ to P must satisfy: $\widetilde{x}_k \leq [\widehat{x}_k]$ **or** $\widetilde{x}_k \geq [\widehat{x}_k] + 1$.
Now solve two mixed integer problems:
P_1: Maximize z subject to $Ax = b$, $x \geq 0$, $x_j \in \mathbb{Z}$ for $j \in I$, **and** $x_k \leq [\widehat{x}_k]$.
P_2: Maximize z subject to $Ax = b$, $x \geq 0$, $x_j \in \mathbb{Z}$ for $j \in I$, **and** $x_k \geq [\widehat{x}_k] + 1$.
The optimal solution to P is the solution with the largest objective value.

Problem: Solving P_1 and P_2 in the same way can lead to lots of branching.



We must try to eliminate as many branches of tree as possible.

Idea 2:

Start by computing optimal solutions $\widehat{x}_1$ and $\widehat{x}_2$ to the linear problems $\widehat{P}_1$ and $\widehat{P}_2$ corresponding to P_1 and P_2.

Suppose we have found some feasible solution $\widetilde{x}_1$ to P_1 such that $z(\widetilde{x}_1) \geq z(\widehat{x}_2)$.

Then by Observation 2 we know that $\widetilde{x}_1$ is as least as good as the optimal solution to P_2. So we don't have to solve P_2 after all! We say that P_2 has been **implicitly enumerated**.

Rooted trees:

A **tree** is a graph without loops, consisting of **nodes** connected by line segments called **branches**. A **rooted tree** is a tree together with a special node called the **root** of the tree. We will draw trees so that they grow downward, i.e. the root is placed at the top, and all branches go down. The nodes at the bottom, for which no branches continue further down, are called **leaves**.

Branch and Bound Algorithm:

Find an optimal solution $\widehat{x}$ to $\widehat{P}$, and let $(\widehat{x}, z(\widehat{x}))$ be the root of a tree. We will gradually build a tree under this root.

Each branch will be labeled by $x_k \leq a$ or $x_k \geq a + 1$ for some $k \in I$ and $a \in \mathbb{Z}$.

Each node of the tree represents the linear problem obtained from $\widehat{P}$ by adding the additional constraints of the branches leading to the node.

A node is labeled by either $(\widehat{x}, z(\widehat{x}))$, if $\widehat{x}$ is an optimal solution of the corresponding linear problem, or by $\emptyset$ if this linear problem has no feasible solutions.

A leaf of the tree is **terminal** if it is labeled by $\emptyset$ or by a feasible solution to the mixed integer problem P. Otherwise the leaf is called **dangling**.

A dangling leaf $(\widehat{x}, z(\widehat{x}))$ is implicitly enumerated if there exists a terminal leaf $(\widetilde{x}, z(\widetilde{x}))$ such that $z(\widehat{x}) \leq z(\widetilde{x})$.

Branching a dangling leaf:

Let $\widehat{x}$ be an optimal solution to the problem of a dangling leaf.

Choose $k \in I$ such that $\widehat{x}_k \notin \mathbb{Z}$. E.g. choose k such that $\text{frac}(\widehat{x}_k)$ is maximal.

Create two branches from the leaf labeled $x_k \leq [\widehat{x}_k]$ and $x_k \geq [\widehat{x}_k] + 1$, and add new leaves at the ends of these branches with the correct labels.

The algorithm:

(1) Label the root of the tree by $(\widehat{x}, z(\widehat{x}))$, where $\widehat{x}$ is an optimal solution to $\widehat{P}$.

(2) If there are dangling leaves that are not implicitly enumerated, **choose** one of them and branch it. Repeat until all dangling leaves are implicitly enumerated.

(3) The optimal solution is the terminal leaf with the largest objective value.

Example:

Maximize $z = 11x_1 + 80x_2 + 13x_3$

subject to

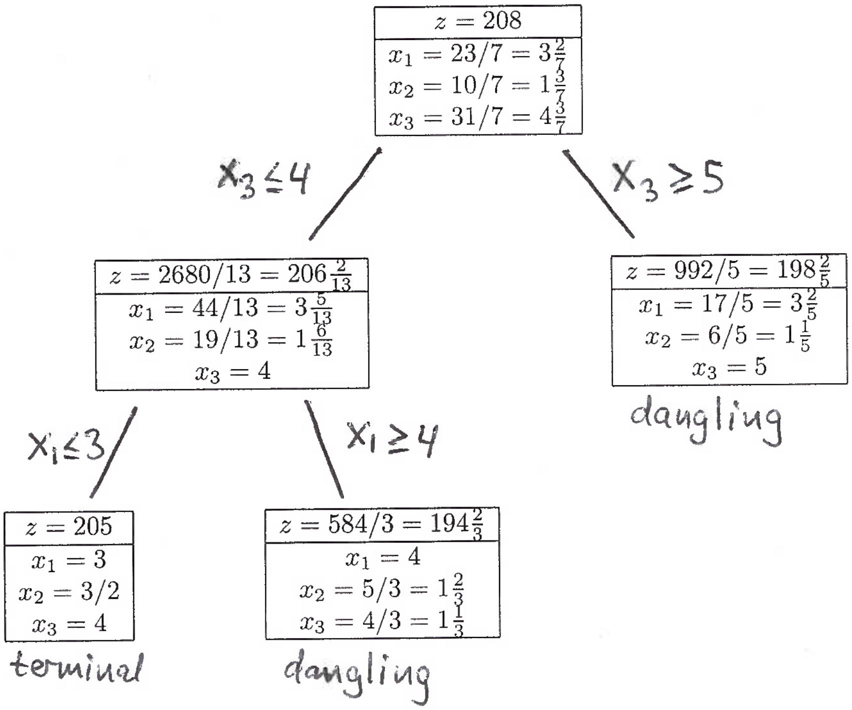
$3x_1 + 4x_2 + x_3 \leq 20$

$x_1 + 10x_2 + x_3 \leq 22$

$x_1 + 8x_2 + 3x_3 \leq 28$

$x \geq 0$ in $\mathbb{R}^3$

$x_1, x_3 \in \mathbb{Z}$.



Optimal solution:

$\widetilde{x} = (3, 3/2, 4)$

$z(\widetilde{x}) = 205$

Linear Opt. 12/8/2015

Last class.

Final Monday 12/21 8–11 AM in ARC-107.

Chapters covered: 1–4.

Skipped sections: "Big M" + computational aspects ×2.

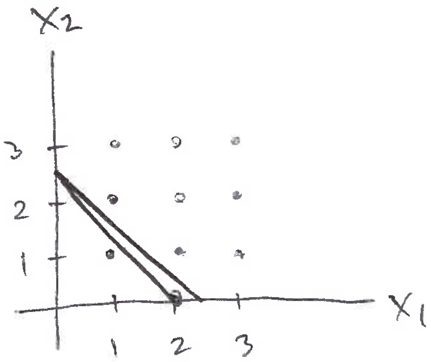
Example:

Maximize $z = 1x_1 + 1x_2$

Subject to $5x_1 + 4x_2 = 10$

$x \geq 0$ in $\mathbb{R}^2$

$x_1, x_2 \in \mathbb{Z}$.



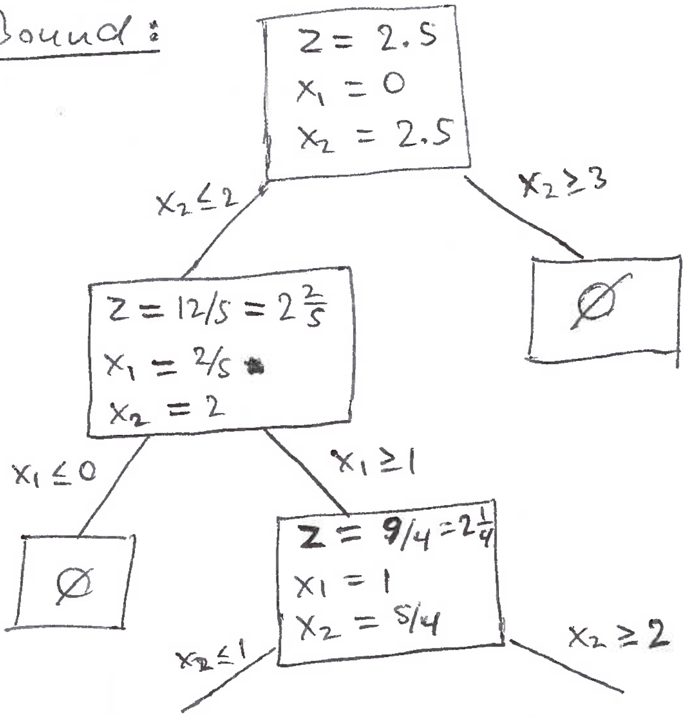
Only one feasible solutions!

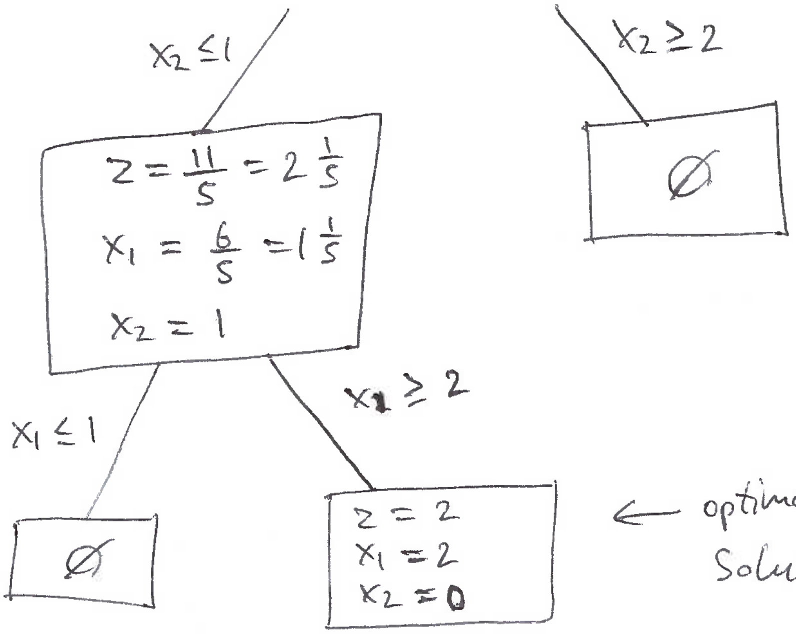
Cutting plane method:

$\frac{5}{4}x_1 + x_2 = \frac{5}{2}$ and $x_1, x_2 \in \mathbb{Z}$.

$\frac{1}{4}x_1 \geq \frac{1}{2}$ (same as $x_1 \geq 2$.)

Branch & Bound:





← optimal (and only feasible) solution.

Primal Problem P

Example Maximize $z = -x_1 + 2x_3 + x_4$

Subject to

$$\begin{aligned} 3x_1 + x_2 - 2x_3 + 3x_4 &= 10 \\ x_1 + x_3 - 2x_4 &\geq -2 \\ x_1 + x_3 + x_4 &\geq 10 \\ x_1 + 2x_3 + 3x_4 &\leq 25 \end{aligned}$$

$x \geq 0$ in $\mathbb{R}^4$

Dual Problem $P^\vee$

Minimize $z^\vee = 10w_1 - 2w_2 + 10w_3 + 25w_4$

Subject to

$$\begin{aligned} 3w_1 - w_2 - w_3 + w_4 &\geq -1 \\ w_1 &\geq 0 \\ -2w_1 - w_2 - w_3 + 2w_4 &\geq 2 \\ 3w_1 + 2w_2 - w_3 + 3w_4 &\geq 1 \end{aligned}$$

$w_2, w_3, w_4 \geq 0$, w_1 unrest.

~~Classical Problem P'~~

Phase I problem ~~Minim~~ Max. $-Y_1$

$$\begin{aligned} 3x_1 + x_2 - 2x_3 + 3x_4 &= 10 \\ -x_1 - x_3 + 2x_4 + u_1 &= 2 \\ + x_1 + x_3 + x_4 - u_2 + Y_1 &= 10 \\ x_1 + 2x_3 + 3x_4 + u_3 &= 25 \end{aligned}$$

opt sol: $\tilde{x} = (x_1, x_2, x_3, x_4) = (0,\ 35,\ \frac{25}{2},\ 0)$.

$z(\tilde{x}) = 25$

Solve dual Problem:

obj ~~test~~ row : $(2, 0, 0, 2, 0, 0, 1, 0)$

$c^T = (-1, 0, 2, 1, 0, 0, 0, 0)$

$\sigma = (1, 0, 2, 3, 0, 0, 1, 0)$

init basic:
x_2 u_1 y_1 u_3

$\hat{w} = (0, 0, 0, 1)$

$\tilde{w} = (0, 0, 0, 1)$

Negate $\hat{w}_3$
(NOT $\hat{w}_2$).