# Genus of complete intersections in spherical varieties

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# Newton polyhedra theory

Let $A \subset \mathbb{Z}^n$ finite.
$L_A := \{f = \sum_{\alpha \in A} c_\alpha x^\alpha\}$ finite dim. subspace of Laurent polynomials.
Fix $A_1, \ldots, A_k \subset \mathbb{Z}^n$ finite. $\Delta_i = \mathrm{conv}(A_i)$.
Pick $f_i \in L_{A_i}$ generic.
$X(f_1, \ldots, f_k) := \{x \in (\mathbb{C}^*)^n \mid f_1(x) = \cdots = f_k(x) = 0\}$.

**Problem (Newton polyhedra theory)**

*Find formulae for discrete geo./top. invariants of $X(f_1, \ldots, f_k)$ in terms of the $\Delta_i$.*

# Euler characteristic of a complete intersection in torus

**Theorem (Khovanskii, 1978)**

*Euler characteristic of* $X(f_1, \dots, f_k) = n!(\prod_{i=1}^{k} \Delta_i(1+\Delta_i)^{-1})_n$, *where:* $\Delta_1^{n_1} \cdots \Delta_k^{n_k} = V(\underbrace{\Delta_1, \dots, \Delta_1}_{n_1}, \cdots, \underbrace{\Delta_k, \dots, \Delta_k}_{n_k})$, $n = n_1 + \cdots + n_k$ *and* $V$ *denotes mixed volume.*

# Euler characteristic of a complete intersection in torus

Case $k = 1$:

**Corollary**

*Euler characteristic of* $X(f) = (-1)^{n-1}\mathrm{vol}_n(\Delta)$.

Case $k = n$:

**Corollary (Bernstein-Kushnirenko, 1978)**

$\#(X_{f_1,\ldots,f_n}) = n! V(\Delta_1, \ldots, \Delta_n)$.

# Genus of a variety

- $X$ compact complex manifold. $\dim(X) = n$.
- $p_g(X) := \dim(H^0(X, \Omega^n)) = h^{n,0}(X)$.
- $p_a(X) := \sum_{p=0}^{n} (-1)^p h^{p,0}(X)$.
- $X$ Riemann surface, $p_g(X) =$ number of holes.
- Birationally invariant, so definition extends to singular varieties also.

# Genus of a hypersurface in torus

$A \subset \mathbb{Z}^n$ finite, $f \in L_A$ generic, $\Delta = \mathrm{conv}(A)$.

**Theorem (Khovanskii, 1978)**

$p_g(X(f)) = N^\circ(\Delta)$, *the number of lattice points in interior of* $\Delta$.

Example: Generic degree $d$ curve in $\mathbb{CP}^2$ is smooth. Genus is $(d-1)(d-2)/2$.

# Genus of a complete intersection in torus

$A_1, \dots, A_k \subset \mathbb{Z}^n$ finite, $f_i \in L_{A_i}$ generic,
$\Delta_i = \operatorname{conv}(A_i)$.

**Theorem (Khovanskii, 1978)**

$p_a(X(f_1, \dots, f_k)) = 1 - \sum_{i_1} N'(\Delta_{i_1}) + \sum_{i_1<i_2} N'(\Delta_{i_1}+\Delta_{i_2}) - \dots + (-1)^k N'(\Delta_1+\dots+\Delta_k)$.

where $N'(\Delta) = (-1)^{\dim(\Delta)} N^\circ(\Delta)$.

# Long exact sequence

$X$ compact complex manifold, $L$ line bundle.
$D_1, \dots, D_k$ smooth transverse hypersurfaces,
$X_m = D_1 \cap \cdots \cap D_m$.
$0 \to \mathcal{O}(X_{m-1}, L \otimes L_m^{-1}) \xrightarrow{i} \mathcal{O}(X_{m-1}, L) \xrightarrow{j}$
$\mathcal{O}(X_m, L) \to 0$.
$0 \to H^0(X_{m-1}, L \otimes L_m^{-1}) \to H^0(X_{m-1}, L) \to$
$H^0(X_m, L) \to \cdots$

# $G$-varieties

$G$ connected reductive group, $X = G/H$ homogen. space.
$L_1, \ldots, L_k$ $G$-line bundles.
$E_i \subset H^0(X, L_i)$ $G$-inv. linear system, $D_i \in E_i$ generic element.

**Problem**

*Find formulae for discrete geo./top. invariants of $X_k = D_1 \cap \cdots \cap D_k$ in terms of convex geo. data.*

# Spherical varieties

**Definition**

$G$-variety $X$ is spherical if a Borel subgroup has a dense orbit.

- $X$ spherical with $G$-line bundle $L \Rightarrow H^0(X, L)$ multiplicity free $G$-module.
- One has a hope of answering the above problem for spherical varieties.

# Examples of spherical varieties

- Toric variety with $G = T$-action.
- $G/P$ with left $G$-action.
- $G$ with $G \times G$-action.
- Space of quadrics $PGL(n, \mathbb{C})/PO(n, \mathbb{C})$ with $G = PGL(n, \mathbb{C})$-action.

# String polytopes of irreducible representation

$G$ conn. reductive group. $\lambda$ dominant weight.

- There is a basis $B_\lambda$ for $V_\lambda$ called *crystal or canonical basis* (Kashiwara-Lusztig).
- Fix a reduced decomposition $\underline{w}_0$ for $w_0$. $\exists$ parametrization $\iota_{\underline{w}_0} : B_\lambda \to \mathbb{Z}^N$ with lattice points in a polytope $\Delta_{\underline{w}_0}(\lambda)$ (*string polytopes* of Littelmann- Berenstein-Zelevinsky).
- (K., 2012) $\iota_{\underline{w}_0}$ coincides with the highest term valuation corr. to the coor. system on a Bott-Samelson variety ass. to $\underline{w}_0$.

# Degree of generalized Plücker embedding

$G/P \hookrightarrow \mathbb{P}(V_\lambda)$. Let $n = \dim(G/P)$.
$L_\lambda$ ass. line bundle.

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**Theorem**

*degree of* $L_\lambda = n!\mathrm{vol}_n(\Delta_{\underline{w}_0}(\lambda))$.

One can think of $\Delta_{\underline{w}_0}(\lambda)$ as Newton polytopes for $G/P$.

# Gelfand-Zetlin polytopes

$G = GL(n, \mathbb{C})$. $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$.

$$\Delta_{GZ}(\lambda) = \begin{array}{cccccc} \lambda_1 & \lambda_2 & \cdots & \cdots & \cdots & \lambda_n \\ & x_{1,n-1} & x_{2,n-1} & \cdots & \cdots & x_{n-1,n-1} \\ & & & \cdots & \cdots & \cdots \\ & & & & x_{1,2} & x_{2,2} \\ & & & & & x_{1,1} \end{array}$$

where $\begin{array}{cc} a & b \\ & c \end{array}$ means $a \leq c \leq b$.

$\Delta_{GZ}(\lambda) \subset \mathbb{R}^{n(n-1)/2}$.

# Moment polytope of a $G$-variety

$G$-variety $X$, $L$ $G$-line bundle.

**Definition (Brion, 1987)**

$$\Delta_{mom}(X, L) = \overline{\{\lambda/k \mid V_\lambda \text{ appears in } H^0(X, L^{\otimes k})\}}.$$

$\Delta_{mom}(X, L) \subset \Lambda^+_{\mathbb{R}}$ convex polytope called *moment polytope*.

$L$ ample $\Rightarrow \Delta_{mom}(X, L)$ coincides with the Kirwan polytope in symplectic geometry (for action of $K$).

# Newton-Okounkov polytope of a spherical variety

$X$ spherical $G$-variety with $G$-line bundle $L$,
$n = \dim(X)$.

**Definition (Okounkov, 1997; Alexeev-Brion, 2004)**

$\Delta_{\underline{w}_0}(X, L) = \bigcup_{\lambda \in \Delta_{mom}(X,L)} (\{\lambda\} \times \Delta_{\underline{w}_0}(\lambda)).$

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**Theorem (Okounkov, Alexeev-Brion)**

*degree of* $L = n!\mathrm{vol}(\Delta_{\underline{w}_0}(X, L)).$

# Polytopes associated to a $G$-invariant linear system

$E \subset H^0(X, L)$ $G$-inv. linear system.

**Definition**

$\Delta_{mom}(X, E) = \overline{\{\lambda/k \mid V_\lambda \text{ appears in } E^k\}}$.

Similar definition for $\Delta_{\underline{w}_0}(X, E)$.

# Genus of a hypersurface in a spherical variety

$X = G/H$ spherical, $n = \dim(X)$.
$E$ globally generated $G$-inv. linear system.
Let $X_1$ generic element of $E$.

**Theorem (K.-Khovanskii, 2015)**

$p_g(X_1) = N^\circ(\Delta_{\underline{w}_0}(X, E))$.

# Genus of a complete intersection in a spherical variety

$E_1, \ldots, E_k$ globally gen. $G$-inv. linear system.
$D_i \in E_i$ generic.
Let $X_k = D_1 \cap \cdots \cap D_k$.

**Theorem (K.-Khovanskii, 2015)**

$p_a(X_k) = 1 - \sum_{i_1} N'(\Delta_{\underline{w}_0}(X, E_{i_1})) + \sum_{i_1<i_2} N'(\Delta_{\underline{w}_0}(X, E_{i_1}E_{i_2})) - \cdots + (-1)^k N'(\Delta_{\underline{w}_0}(X, E_1 \cdots E_k))$.

# Genus of complete intersections in flag variety

$G = GL(n, \mathbb{C})$, $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$.

**Corollary**

$p_a(X_k) = 1 - \sum_{i_1} N'(\Delta_{GZ}(\lambda_{i_1})) + \sum_{i_1 < i_2} N'(\Delta_{GZ}(\lambda_{i_1}) + \Delta_{GZ}(\lambda_{i_2})) - \cdots + (-1)^k N'(\Delta_{GZ}(\lambda_1) + \cdots + \Delta_{GZ}(\lambda_k))$.

# Remarks

Ingredients of the proof:

- Equivariant resolution of singularities for spherical varieties.
- Ehrhart-Macdonald reciprocity and more generally Khovanskii-Pukhlikov theory.
- Vanishing of higher cohomologies of $G$-line bundles for spherical varieties (Brion).

We also obtain conditions for when $X_k$ is irreducible.

# Thank you!