Combinatorial Aspects of Schubert Calculus in Elliptic Cohomology

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Joint work with Kirill Zainoulline (Univ. of Ottawa)

arxiv:1408.5952, 1508.03134, and a forthcoming paper with Changlong Zhong (SUNY Albany)

Elliptic cohomology

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- cohomology for $\mu_1=\mu_2=0$;
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- connective K-theory for $\mu_2=0$.

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Weyl group $W = \langle s_\alpha \,:\, \alpha \in \Phi \rangle = \langle s_i \,:\, i = 1, \ldots, r \rangle$.

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Definition. For all $i \in I$, we define in Q_W the Demazure and push-pull element:

$$X_i := \frac{1}{x_{\alpha_i}}(\delta_{s_i} - 1),$$
$$Y_i := (1 + \delta_{s_i})\frac{1}{x_{-\alpha_i}}.$$

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Fact. Fixing a reduced word $I_w = (i_1, \ldots, i_l)$ for each $w \in W$, $\mathbf{D}_F$ has two distinguished bases:

$$X_{I_w} := X_{i_1} \ldots X_{i_l}\,, \qquad Y_{I_w} := Y_{i_1} \ldots Y_{i_l}\,.$$

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(c) If $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle = -1$ (type A_2), then we have twisted braid relations:

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(d) More involved twisted braid relations in types B_2 and G_2.

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We view the elements of $\bigoplus_{w\in W} S$ as $(f_w)_{w\in W}$, or as functions $f : W \to S$.

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Main tool: the Kazhdan-Lusztig basis of a corresponding Hecke algebra.

Formal root polynomials and their properties

Let $I_w = (i_1, \ldots, i_l)$, which induces a reflection order on $\Phi^+ \cap w\Phi^-$, namely

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Definition. The formal Y-root polynomial is

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Similarly, the formal X-root polynomial is

$$\mathcal{R}^X_{I_w} := \prod_{k=1}^{l} h^X_{i_k}(\beta_k), \qquad \text{where } h^X_i(\beta) = 1 + y_{-\beta} X_i\,.$$

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In particular, if α_i, α_j are the simple roots of a root system of type A_2, then

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Corollary. (L.-Zainoulline) The root polynomial $\mathcal{R}_{I_w}$ does not depend on the choice of I_w if the underlying formal group law $F(x,y)$ is the hyperbolic one; so we can write $\mathcal{R}_w$ instead.

Applications of root polynomials: hyperbolic case

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Note. The ordinary cohomology b-coefficients feature prominently in the work of Kostant-Kumar, as they encode information about the singularities of Schubert varieties.

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Theorem. (L.-Zainoulline) In the hyperbolic case, we have in S:

$$b^Y_{w,I_v} = *(\theta_w\, K^Y(I_v, w))\,, \qquad b^X_{w,I_v} = *(K^X(I_v, w))\,,$$

where $\theta_w \in S$ is called the "normalizing parameter".

Corollaries for cohomology, K-theory and connective K-theory

We derive the following as immediate corollaries of our previous result:

- The formulas of Andersen-Jantzen-Soergel/Billey and Graham-Willems for the localization of Schubert classes and their duals at torus fixed points, in ordinary cohomology and K-theory, respectively.

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- Similar formulas in connective K-theory.
- Duality in connective K-theory (does not follow from the Kostant-Kumar duality in ordinary K-theory; we use duality result for generalized cohomology of Calmès-Zainoulline-Zhong).

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Problem. Define a Schubert basis, i.e., classes which are independent of a reduced word.

The standard topological approach only works if X_w is smooth, and

$$[X_w]_v = \frac{\prod_{\beta \in \Phi^+} y_{-\beta}}{\prod_{\substack{\beta \in \Phi^+ \\ s_\beta v \leq w}} y_{-\beta}}\,, \quad \text{for } v \leq w; \text{ otherwise } [X_w]_v = 0\,.$$

A Schubert basis via the Kazhdan-Lusztig basis

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Definition. Consider the element (Kazhdan-Lusztig Schubert class) $\mathfrak{S}_w$ in $Ell_T^*(G/B)$ given by

$$(t + t^{-1})^{-\ell(w)}\, \Gamma_{w^{-1}}(\zeta_\emptyset)\,.$$

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(3) (with C. Zhong) in all types for $w = w_\circ$, and the parabolic case too, when the class of the flag variety is 1.

A positivity conjecture

Recall that

$$\mu_1 = 1\,, \quad \mu_2 = -(t + t^{-1})^{-2}\,, \quad u := -\mu_2$$

in the hyperbolic formal group law

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- c is a positive integer,
- m is the degree of the monomial,
- N is the number of positive roots,
- $N - \ell(v) \leq k \leq m$,
- $m - k$ is even.

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(3) More explicit formulas, e.g., in the maximal parabolic case (type A Grassmannian etc.).

Planned workshop: Equivariant Generalized Schubert Calculus and Its Applications

Organizers: Cristian Lenart, Kirill Zainoulline and Changlong Zhong

Location: University of Ottawa

Proposed dates: April 28–May 1, 2016