(Equivariant) Chern-Schwartz-MacPherson classes

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Let X be a compact manifold, T_X tangent bundle, with Chern class

$$c(T_X) = 1 + c_1(T_X) + \ldots + c_n(T_X).$$

Gauss-Bonnet Theorem:

$$c_n(T_X) \cap [X] = \chi(X)$$

the topological Euler characteristic of X.
Question: What happens if X is singular ?

Constructible functions

Let X be an algebraic variety. Constructible functions:

$$\mathcal{F}(X) = \{\sum c_i \mathbb{1}_{V_i} : c_i \in \mathbb{Z}, V_i \subset X \text{ constructible } \}.$$

If $f : X \to Y$ is a proper map, define a push-forward

$$f_* : \mathcal{F}(X) \to \mathcal{F}(Y); \quad f_*(\mathbb{1}_V)(y) = \chi(f^{-1}(y) \cap V).$$

Chern-Schwartz-MacPherson classes

Theorem (Deligne - Grothendieck Conjecture; MacPherson '74, M. H. Schwartz)

There exists a unique natural transformation $c_* : \mathcal{F}(X) \to H_*(X)$ *such that:*

1. *If* X *is projective, non-singular,* $c_*(1\!\!1_X) = c(T_X) \cap [X]$.
2. c_* *is functorial with respect to proper push-forwards* $f : X \to Y$:

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{c_*} & H_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathcal{F}(Y) & \xrightarrow{c_*} & H_*(Y) \end{array}$$

$$c_{\mathrm{SM}}(X) := c_*(1\!\!1_X)$$

is the Chern-Schwartz- MacPherson class.

Aluffi's method

Let X closed, $\pi : Z \to X$ be a resolution of singularities, and $D_X \subset X$ a divisor such that

$$\pi^{-1}(D_X) = D := D_1 \cup \ldots \cup D_k$$

is simple normal crossing (SNC) and $\pi : Z \setminus D \simeq X \setminus D_X$. Then

$$\begin{aligned} c_{\mathrm{SM}}(X \setminus D_X) &= \pi_*(c_{\mathrm{SM}}(Z \setminus D)) \\ &= \pi_*(c_{\mathrm{SM}}(Z) - c_{\mathrm{SM}}(D)) \\ &= \pi_*\left(\frac{c(T_Z)}{(1+D_1)(1+D_2)\ldots(1+D_k)} \cap [Z]\right). \end{aligned}$$

Goal: Apply this to a (Schubert variety $\setminus$ boundary divisor) and a Bott-Samelson resolution.

Lie and Schubert data

G - complex simple Lie group and $T \subset B \subset G$ (torus $\subset$ Borel $\subset$ G).
E.g. $G = \mathrm{SL}_n(\mathbb{C})$ and $B =$ upper triangular matrices.
$W := N_G(T)/T$ - the Weyl group.
$\ell : W \to \mathbb{N}$ - length function.
s_i - simple reflections; w_0 - longest element in W.
G/B - generalized flag manifold; e.g. $\mathrm{Fl}(n) = \{F_1 \subset \ldots \subset F_n = \mathbb{C}^n\}$.
$X(w)^o := BwB/B$ - Schubert cell.
$X(w) := \overline{BwB/B}$ - Schubert variety.

$$\dim_{\mathbb{C}} X(w) = \ell(w); \quad \ell(w_0) = \dim G/B.$$

$\partial X(w) := X(w) \setminus X(w)^o$ - boundary divisor.

Bott-Samelson varieties and CSM classes

To any reduced decomposition $w = s_{i_1} \dots s_{i_k}$ one can define the Bott-Samelson-(Demazure-Hansen) variety $Z(w)$ inductively as a tower of $\mathbb{P}^1$-bundles. It comes equipped with

$$\pi : Z(w) \to X(w)$$

proper, birational such that

$$\pi^{-1}(\partial X(w)) = D := D_1 \cup \dots \cup D_k$$

is a SNC divisor and $\pi : Z(w) \setminus D \simeq X(w) \setminus \partial X(w)$.

Corollary (Aluffi)

$$c_{SM}(X(w)^o) = \pi_*\left(\frac{c(T_{Z(w)})}{(1+D_1)(1+D_2)\dots(1+D_k)} \cap [Z(w)]\right).$$

Examples

Recall that $H^*(G/B) = \oplus_{w \in W} \mathbb{Z}[X(w)]$. Corollary immediately implies:

$$c_{SM}(X(w)^o) = \sum_{v \leq w} c(w;v)[X(v)] = 1 \cdot [X(w)] + \ldots + 1 \cdot [pt].$$

1. $G/B = \mathbb{P}^1$. Then
$$c_{SM}(\mathbb{P}^1) = c(T_{\mathbb{P}^1}) \cap [\mathbb{P}^1] = [\mathbb{P}^1] + 2[pt].$$
2. $c_{SM}[pt] = [pt]$ thus
$$c_{SM}(\mathbb{A}^1) = c_{SM}(\mathbb{P}^1)) - c_{SM}([pt]) = [\mathbb{P}^1] + [pt].$$

Operators on $H^*(G/B)$

The BGG operator: let $P_k \subset G$ minimal parabolic.

$$\begin{array}{ccc} G/B \times^{G/P_k} G/B & \xrightarrow{pr_1} & G/B \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle p} \\ G/B & \xrightarrow{p} & G/P_k \end{array}$$

$$\partial_k = (pr_2)_*(pr_1)^* : H^*(G/B) \to H^{*-2}(G/B).$$

Right Weyl group action: Let $s_k \in W$. Since $G/B \simeq_{hom} G/T$, right multiplication induces

$$s_k : H^*(G/B) \to H^*(G/B) \quad \text{automorphism.}$$

Alternatively, using Chevalley rule

$$s_k = id - c_1(\mathcal{L}_{-\alpha_k})\partial_k.$$

Formulas for left/right W-actions on $H^*_T(G/B)$ found by: Peterson, Knutson, Tymoczko,....

A Demazure-Lusztig type operator

Define $\mathcal{T}_k := \partial_k - s_k$.

Note: This operator is a specialization of an operator which appears in the study of a degenerate affine Hecke algebra, in relation to the Steinberg variety in $T^*_{G/B} \times T^*_{G/B}$.

Lemma

The operators $\mathcal{T}_k$ satisfy the following properties:

1. *(commutativity) E.g. in type A, $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$ if $|i-j| \geq 2$;*
2. *(braid relations) E.g. in type A: $\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}$;*
3. *(square) $\mathcal{T}_i^2 = id$.*
4. *(Schubert action): $\mathcal{T}_k([X(w)]) =$*

$$\begin{cases} -[X(w)] & \text{if } \ell(ws_k) < \ell(w) \\ [X(ws_k)] + [X(w)] + \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)] & \text{if } \ell(ws_k) > \ell(w) \end{cases}$$

where $\beta > 0$, $\beta \neq \alpha_k$ and $\ell(ws_k s_\beta) = \ell(w)$.

CSM classes of Schubert cells

Theorem (Aluffi-M.)

1. *Let $w \in W$ be a Weyl group element, and $X(w)^o \subset G/B$ the Schubert cell. Then*

$$\mathcal{T}_k(c_{SM}(X(w)^o)) = c_{SM}(X(ws_k)^o).$$

2. *Let $P \subset G$ be any parabolic subgroup and $pr : G/B \to G/P$ be the projection. Then*

$$pr_*(c_{SM}(X(w)^o)) = c_{SM}(X(wW_P)^o)$$

where $W_P \leq W$ the the subgroup generated by the reflections in P.

Positivity

$\mathrm{Fl}(3) = \{F_1 \subset F_2 \subset \mathbb{C}^3\}$ - the flag variety.
$\dim \mathrm{Fl}(3) = \ell(w_0) = \ell(s_1 s_2 s_1) = 3$.

$$c_{SM}(\mathrm{Fl}(3)^o) = [\mathrm{Fl}(3)] + [X(s_2 s_1] + [X(s_1 s_2)] + 2[X(s_1)] + 2[X(s_2)] + [pt]$$

Conjecture

The coefficients $c(w;u) > 0$ for any $u \leq w$.

- J. Huh proved the conjecture in the case $G/P =$ Grassmannian.
- Positivity has been checked for $\mathrm{Fl}(n)$, $n \leq 7$.
- We proved the conjecture in some cases: $\ell(w) - \ell(u) \leq 1$ or if w has a reduced decomposition into distinct reflections.

Equivariant case

T. Ohmoto defined an equivariant version of MacPherson's transformation:

$$c_*^T : \mathcal{F}_T(X) \to H_*^T(X)$$

which satisfies functoriality and

$$c_*^T(1\!\!1_X) = c^T(T_X) \cap [X]_T \text{ if } X \text{ is projective, nonsingular.}$$

A. Weber proved properties of localizations of CSM classes, and Rimányi-Varchenko used these and Maulik-Okounkov stable envelopes to obtain localization formulas for CSM classes ${c_{\mathrm{SM}}}^T(X(w)^o)_{|u}$.

Theorem (Aluffi - M.)

Let $\mathcal{T}_k^T := \partial_k - s_k$*. Then*

$$\mathcal{T}_k^T({c_{SM}}^T(X(w)^o)) = {c_{SM}}^T(X(ws_k)^o).$$

The equivariant operator

The operator $\mathcal{T}_k^T$ acts almost as $\mathcal{T}_k$ on Schubert classes: $\mathcal{T}_k([X(w)]) =$

$$\begin{cases} -[X(w)] \\ (1 + w(\alpha_k))[X(ws_k)] + [X(w)] + \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)] \end{cases}$$

where branches are as before. The $(c(w;u))$ matrix for cells in $\mathrm{Fl}(3)$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1+\alpha_2 & 0 & 2+\alpha_1+\alpha_2 & 1+\alpha_2 & 2+\alpha_1+\alpha_2 \\ 0 & 0 & 1+\alpha_1 & 1+\alpha_1 & 2+\alpha_1+\alpha_2 & 2+\alpha_1+\alpha_2 \\ 0 & 0 & 0 & (1+\alpha_1)(1+\alpha_1+\alpha_2) & 0 & (1+\alpha_1)(1+\alpha_1+\alpha_2) \\ 0 & 0 & 0 & 0 & (1+\alpha_2)(1+\alpha_1+\alpha_2) & (1+\alpha_2)(1+\alpha_1+\alpha_2) \\ 0 & 0 & 0 & 0 & 0 & (1+\alpha_1)(1+\alpha_2)(1+\alpha_1+\alpha_2) \end{pmatrix}$$

Read on columns!

Conjecture (Equivariant positivity)

For any $u \leq w$, the coefficients $c(w;u)$ are polynomials with non-negative coefficients in simple roots α_i.

Further connections

- Let $\iota : G/B \to T^*_{G/B}$ be the zero section and let $Stab_+(w) \in H^*_{T\times\mathbb{C}^*}(T^*_{G/B})$ be the stable envelope. Changjian Su used the operator $\mathcal{T}_k^T$ to prove:

$$\iota^* Stab_+(w)_{|\hbar=1} = \pm P.D.\ {c_{\mathrm{SM}}}^T(X(w)^o).$$

- Let $L_{\mathbb{C}^*}(T^*_{G/B})$ be the group of Lagrangian cycles. Ginzburg proved that MacPherson's map c_* factors as

$$\mathcal{F}(G/B) \xrightarrow{\simeq} L_{\mathbb{C}^*}(T^*_{G/B}) \xrightarrow{c_*^{Gi}} H_*(G/B)$$

  Then (C. Su - M., J. Schürmann):

$$c_*^{Gi}(Stab_+(w)) = \pm c_{\mathrm{SM}}(X(w)^o).$$

- Seung-Jin Lee: in type A, the coefficients $c(w;u)$ coincide with certain specializations in Fomin-Kirillov algebra.

THANK YOU!