Knutson-Vakil puzzles compute equivariant K-theory of Grassmannians

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Joint with Alexander Yong (UIUC)
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Setup

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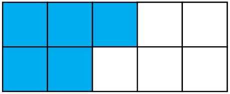

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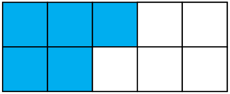

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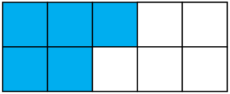

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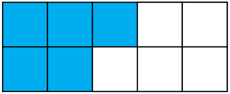

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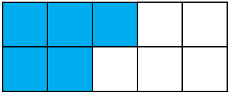

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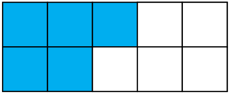

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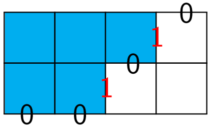
- Schubert classes σ_λ give a $\mathbb{Z}$-basis of $H^\star(X)$
- $$c^{\nu}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$$

Cohomological puzzles

- Partitions inside $k \times (n-k) \longleftrightarrow$ binary strings of length n with k 1's



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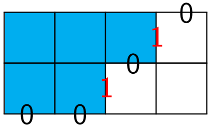

- Let $\Delta_{\lambda,\mu,\nu}$ be an equilateral triangle of side length n with the boundary labeled by
  - λ as read $\nearrow$ along the left side;
  - μ as read $\searrow$ along the right side; and
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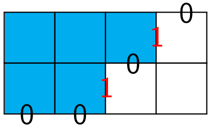

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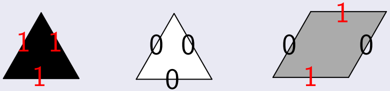
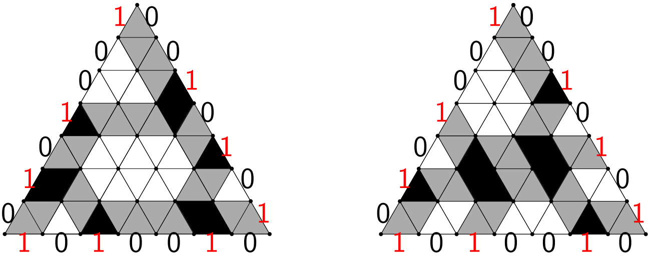
Theorem (A. Knutson–T. Tao 1999)

$c_{\lambda,\mu,\nu}$ counts tilings of $\Delta_{\lambda,\mu,\nu}$ by the following puzzle pieces:



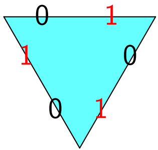
Example puzzle calculation

$c^{(4,3,1)}_{(3,2),(2,1)} = 2$ is calculated by the tilings:



Puzzles in richer cohomology theories

- In K-**theory**, structure coefficients are computed by puzzles with an extra (non-rotatable) piece due to A. Buch:



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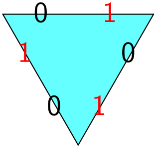

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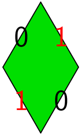
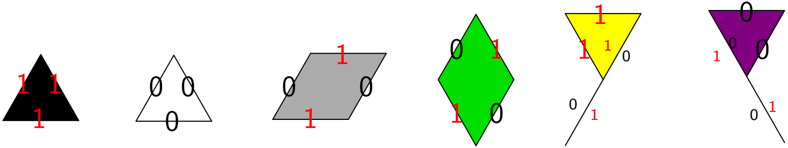
- In T-**equivariant cohomology**, structure coefficients are computed by puzzles with an extra (non-rotatable) piece due to A. Knutson–T. Tao:



  It has weight $t_i - t_j$, where i, j depend on the location.

The Knutson-Vakil conjecture



- The equivariant green rhombus now has weight $1 - \frac{t_i}{t_j}$
- The purple and yellow gashed triangles have weight -1

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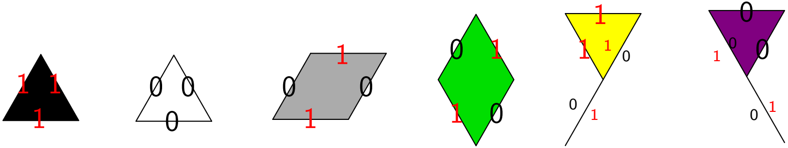

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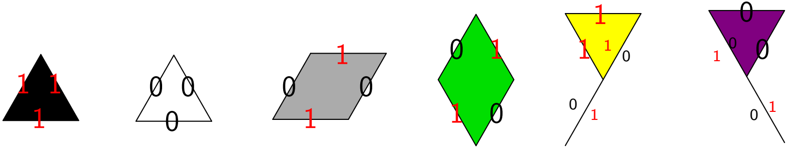

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- There is a 'non-local' requirement for using :

  "It may only be placed (when completing the puzzle from top to bottom and left to right as usual) if the edges to its right are a (possibly empty) series of horizontal 0's followed by a 1"

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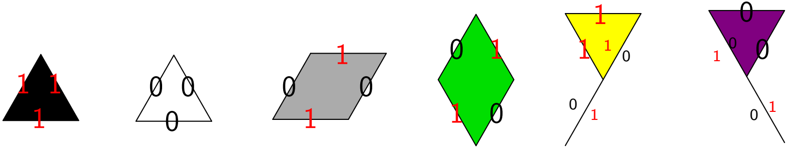

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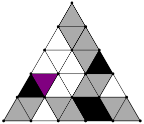
Conjecture (A. Knutson–R. Vakil 2005)

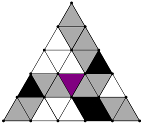
The T-equivariant K-theory coefficient $c^{\nu}_{\lambda,\mu}$ is the weighted count of all such puzzle fillings of $\Delta_{\lambda,\mu,\nu}$.

Counterexample

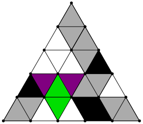
For $c_{(2),(1)}^{(3,1)}$ for $\mathrm{Gr}_2(\mathbb{C}^5)$, there are six KV-puzzles $P_1, P_2, \ldots, P_6$.



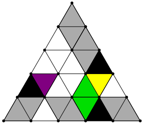
$\mathrm{wt}(P_1) = -1$



$\mathrm{wt}(P_2) = -1$



$\mathrm{wt}(P_3) = (-1)^2(1 - \frac{t_3}{t_4})$



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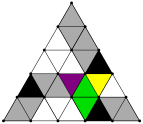
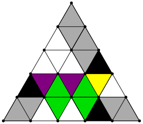
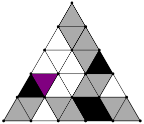


$\mathrm{wt}(P_5) = (-1)^2(1 - \frac{t_2}{t_3})$



$\mathrm{wt}(P_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$

For $c_{\square\square,\square}^{(3,1)}$ for $\mathrm{Gr}_2(\mathbb{C}^5)$, there are six KV-puzzles $P_1, P_2, \ldots, P_6$.



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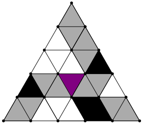

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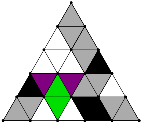

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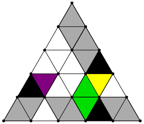

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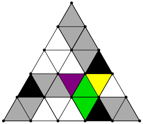

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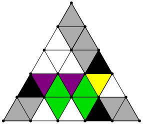

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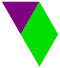
- However
  $$c_{\square\square,\square}^{(3,1)} = -(1 - \frac{t_2}{t_4}) = \mathrm{wt}(P_2) + \mathrm{wt}(P_3) + \mathrm{wt}(P_5) + \mathrm{wt}(P_6)$$
- Knutson-Vakil conjecture is false

Modified KV-puzzles compute $c^{\nu}_{\lambda,\mu}$

- But the Knutson-Vakil conjecture is almost correct
- Replace the complicated 'non-local' condition on  with the condition that  only appears in the combination pieces

 and 

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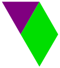
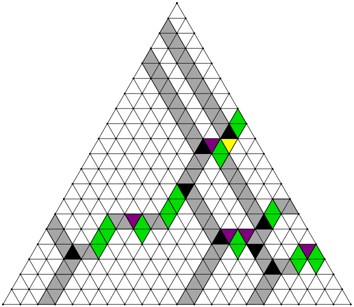
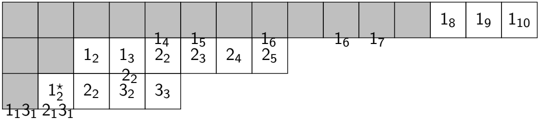
 and 

Theorem (P.–Yong 2015)

The T*-equivariant* K*-theory coefficient* $c_{\lambda,\mu}^{\nu}$ *is the weighted count of all modified KV-puzzles with boundary* $\Delta_{\lambda,\mu,\nu}$*.*

Bijection to genomic tableaux





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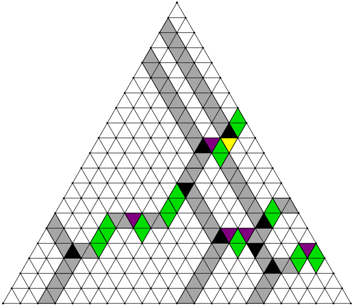

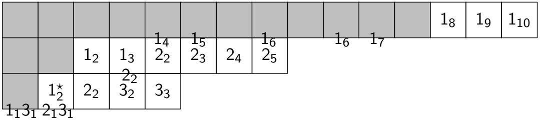

THANK YOU!